On the structure of residuated po-semigroups as models of Lambek Calculus and MLL

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Outline

- Involutive po-monoids as models of multiplicative linear logic
- Examples: binary relations, pointed groups, po-groups
- Partially ordered algebras
- Residuated po-semigroups as models of Lambek Calculus
- The structure of finite ipo-monoids with a Sugihara component
- One more thing (joint work with S. Santschi)

Involutive po-monoids

An **involutive po-monoid** (ipo-monoid) is an algebraic structure $(A, \leq, \cdot, \sim, -, 0)$ such that $(A, \leq)$ is a poset, $\cdot$ is **associative** and

$$\text{(lin)} \qquad x \leq y \iff x \cdot {\sim}y \leq 0 \iff -y \cdot x \leq 0$$

Even without associativity (lin) has some interesting consequences:

Lemma

$-, \sim$ form a dual Galois connection.

Proof.

$-x \leq y \iff -x \cdot {\sim}y \leq 0 \iff {\sim}y \leq x.$ □

Hence $-, \sim$ are order reversing, $\quad -{\sim}x \leq x, \qquad {\sim}{-}x \leq x,$

${\sim}{-}{\sim}x = {\sim}x \quad$ and $\quad -{\sim}{-}x = -x.$

But wait, there is more!

The linear negations are involutive

Lemma

$\sim{-x} = x = -{\sim x}$, *i.e.*, $\sim, -$ *are involutions.*

Proof.

$x \leq -{\sim x} \iff x \cdot {\sim}{-}{\sim x} \leq 0 \iff x \cdot {\sim x} \leq 0 \iff x \leq x.$ □

For the next few results we use associativity.

Lemma

${\sim 0} = -0$ *is an identity element.*

Proof.

$-0 \cdot x \leq y \iff -0 \cdot x \cdot {\sim y} \leq 0 \iff x \cdot {\sim y} \leq 0 \iff x \leq y$. Hence $-0 \cdot x = x$ and similarly $x \cdot {\sim 0} = x$. Therefore ${\sim 0} = -0 \cdot {\sim 0} = -0$. □

Naturally ${\sim 0}$ is denoted by 1.

Multiplication is residuated, hence order-preserving

Lemma

$x{\cdot}y \le z \iff x \le -(y \cdot {\sim}z) \iff y \le {\sim}(-z \cdot x).$

Proof.

$x{\cdot}y \le z \iff (-{\sim}x){\cdot}y{\cdot}{\sim}z \le 0 \iff y{\cdot}{\sim}z \le {\sim}x \iff x \le -(y{\cdot}{\sim}z).$
Similarly
$x{\cdot}y \le z \iff -z{\cdot}x{\cdot}({\sim}-y) \le 0 \iff -z{\cdot}x \le -y \iff y \le {\sim}(-z{\cdot}x).$ □

Therefore $z/y = -(y \cdot {\sim}z)$ and $x\backslash z = {\sim}(-z \cdot x)$.

Lemma

$x \le y \implies xz \le yz$ *and* $zx \le zy$.

Proof.

Follows from residuation: $yz \le yz \implies x \le y \le yz/z \implies xz \le yz.$ □

Algebraic models of multiplicative linear logic (MLL)

The previous results show that ipo-monoids are po-algebraic models of (noncommutative, noncyclic) **multiplicative linear logic** (MLL).

Usually MLL is presented as a sequent calulus. Here $\leq$ is $\vdash$.

$x + y = {\sim}(-y \cdot -x) = -({\sim}y \cdot {\sim}x)$.

${}^{\perp}x = -x, \qquad x^{\perp} = {\sim}x$.

Examples: binary relations, pointed groups, po-groups

Any set of binary relations $A \subseteq \mathcal{P}(X^2)$ closed under composition and **complement-inverse** $\sim R = -R = X^2 \setminus R^{-1}$.

A **pointed group** $(A, \cdot, ^{-1}, 1, 0)$ is a group with an (arbitrary) constant 0.

To obtain an ipo-monoid, let $\leq$ be equality and define $\sim x = x^{-1} \cdot 0$ and $-x = 0 \cdot x^{-1}$.

A **partially ordered group** is a structure $\mathbf{A} = (A, \leq, \cdot, ^{-1}, 1)$ such that

1. $(A, \leq)$ is a poset
2. $(A, \cdot, ^{-1}, 1)$ is a group and
3. multiplication is order preserving (hence $^{-1}$ is order reversing).

$\implies 0 = 1^{-1} = 1$ and **cyclicity** holds: $\sim x = -x = x^{-1}$.

Examples: $(\mathbb{R}, \leq, +, -, 0)$, $(\mathbb{Q}^+, \leq, \cdot, ^{-1}, 1)$, **all** groups (with $\leq$ is $=$)

More examples: Sugihara ipo-monoids

Let S_{2n} be a $2n$-element chain $a_n < \cdots < a_2 < 0 < 1 < b_2 < \cdots < b_n$ and

S_{2n-1} a $(2n-1)$-element chain $a_n < \cdots < a_2 < 0 = 1 < b_2 < \cdots < b_n$.

Define $a_1 = 0, \quad b_1 = 1, \quad {\sim}a_i = -a_i = b_i, \quad {\sim}b_i = -b_i = a_i,$

$$a_i \cdot a_j = a_{\max(i,j)}, \quad b_i \cdot b_j = b_{\max(i,j)}, \quad a_i \cdot b_j = b_j \cdot a_i = \begin{cases} b_j & \text{if } i < j \\ a_i & \text{otherwise} \end{cases}$$

$(S_m, \leq, \cdot, \sim, -, 0)$ is the **m-element Sugihara ipo-monoid** (no $\wedge, \vee$)

$S_2 = \mathbf{2}$ the two-element **Boolean algebra**

S_3 is also known as the **Gaifman-Pratt ipo-monoid**

Models from products of groups and ipo-chains

Let G be a group with subgroup H.

Define $A = H \times S_2 \ \cup \ (G \setminus H) \times S_1$ with

$(g, a) \cdot (g', a') = (gg', aa')$ and $\sim(g, a) = (g^{-1}, \sim a) = -(g, a)$.

Then A is an ipo-monoid.

Our structure theorem says that any finite ipo-monoid where the connected component with 1 is isomorphic to S_m will be a **union of chains of length $\leq m$** with its monoid structure uniquely determined by a sequence of **nested subgroups of G**.

This is a generalization of closely related results by Zhuang [2023], Galatos and Zhuang [2024] on **unilinear residuated lattices**.

Partially ordered algebras

A **po-algebra** is a partially ordered set with operations that are either order-preserving **or order-reversing** in each argument.

A **variety of po-algebras** is a class of similar po-algebras defined by equations **or inequations** [Pigozzi 2004].

A **residuated partially ordered magma** or **rpo-magma** $\mathbf{A} = (A, \leq, \cdot, \backslash, /)$ is a partially-ordered set $(A, \leq)$ with a binary operation $\cdot$ and two **residuals** that satisfy for all $x, y, z \in A$

$$\text{(res)} \qquad xy \leq z \iff x \leq z/y \iff y \leq x\backslash z$$

The operation $x \cdot y$ is usually written xy.

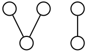
Residuated po-magmas are a po-variety

Residuation ensures that x/y and $y\backslash x$ are order-preserving in the numerator (x position) and **order-reversing** in the denominator.

xy is order-preserving in both arguments.

(res) is equivalent to $x \le xy/y$, $(z/y)y \le z$, $y \le x\backslash xy$, $x(x\backslash z) \le z$ hence rpo-magmas are a variety of po-algebras.

Although rpo-magmas are very general, (res) imposes restrictions on the posets that can occur.

E.g. could  be the poset of a rpo-magma?

Lemma

For rpo-magmas, if $a, b \leq c$ *then* $(a/(a\backslash c))((c/(a\backslash c))\backslash b) \leq a, b$.

Proof.

$$
\begin{array}{lll}
\text{Assume } a \leq c. & \text{Then} & a/(a\backslash c) \leq c/(a\backslash c) \\
\Longrightarrow & & (c/(a\backslash c))\backslash b \leq (a/(a\backslash c))\backslash b \\
\Longleftrightarrow & & (a/(a\backslash c))((c/(a\backslash c))\backslash b) \leq b \\
\text{Assume } b \leq c. & \text{Then} & a\backslash c \leq a\backslash c \\
\Longleftrightarrow & & a(a\backslash c) \leq c \\
\Longleftrightarrow & & a \leq c/(a\backslash c) \\
\Longrightarrow & & (c/(a\backslash c))\backslash b \leq a\backslash b \leq a\backslash c \\
\Longrightarrow & & a/(a\backslash c) \leq a/((c/(a\backslash c))\backslash b) \\
\Longleftrightarrow & & (a/(a\backslash c))((c/(a\backslash c))\backslash b) \leq a
\end{array}
$$

□

Finite rpo-magmas have bounded components

Lemma

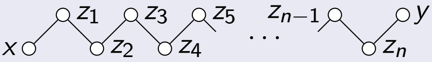
In any rpo-magma, if $d \le a, b$ then
$a, b \le d/(((a\backslash d)/(a\backslash(d\backslash d)))((d\backslash d)/(a\backslash(d\backslash d)))\backslash(b\backslash d)))$

Theorem

In an rpo-magma every connected component of $\le$ is up-directed and down-directed, hence for **finite** *rpo-magmas every connected component is* **bounded***.*

Proof.

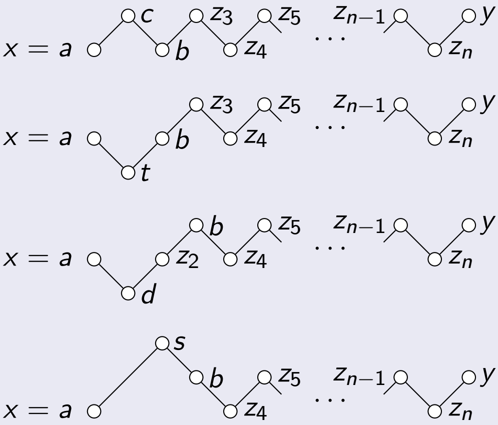
Two elements x, y in a poset are connected iff there exists a zigzag



We need to find an upper and a lower bound of x, y.

□

Proof (continued).



Use induction to get an upper and lower bound of x, y in n steps.

What posets are possible for rpo-magmas?

In a poset define $x \rightsquigarrow y$ if x and y are in the same connected component.

Theorem

For any po-algebra the relation $\rightsquigarrow$ is a congruence.

For a rpo-magma A the quotient algebra $A/\rightsquigarrow$ is a **quasigroup** *with $\leq$ as equality, i.e., $xy = z \iff x = z/y \iff y = x\backslash z$.*

Conversely, from any quasigroup Q and a pairwise disjoint family of **bounded** *posets A_q for $q \in Q$, one can construct an rpo-magma with poset $\bigcup_{q \in Q} A_q$.*

E.g. for a quasigroup Q and $x \in A_p$, $y \in A_q$ define

$$x \cdot y = \bot_{pq}, \qquad x\backslash y = \top_{p\backslash q}, \qquad x/y = \top_{p/q}.$$

Residuated po-semigroups

A **rpo-semigroup** or **Lambek algebra** is a rpo-magma where $\cdot$ is associative. These are algebraic models of Lambek Calculus.

Lemma

If the poset is an antichain then a rpo-semigroup is a **group***.*

Proof.

An associative quasigroup is a group: $x = y\backslash yx = y(y\backslash x)$. Hence $x = x(x\backslash x) \Rightarrow xy = x(x\backslash x)y \Rightarrow y = x\backslash xy = (x\backslash x)y \Rightarrow y/y = x\backslash x$. So $x\backslash x = e$ is constant, $xe = x$ and $ey = y$, i.e., e is an identity.

Now $(e/x)x = e \Rightarrow x(e/x)x = x \Rightarrow x(e/x) = x/x = e$, so $x^{-1} = e/x$. $\square$

$\Rightarrow$ For any rpo-semigroup A the quotient $A/\!\!\rightsquigarrow$ is a group.

Ipo-monoids where $0 \le 1$

Lemma

In an ipo-monoid with $0 \le 1$, for any n, $({\sim}x)^n \le {\sim}x^n$ and $(-x)^n \le -x^n$.

Proof.

By induction, assume $({\sim}x)^n \le {\sim}x^n$. Then $x^n({\sim}x)^n \le 0 \le 1$.

Now $x \cdot x^n({\sim}x)^n \cdot {\sim}x \le x \cdot {\sim}x \le 0$.

Hence $x^{n+1}({\sim}x)^{n+1} \le 0$ and therefore $({\sim}x)^{n+1} \le {\sim}x^{n+1}$. □

For a rpo-monoid A, the connected components are denoted C_g for $g \in A/\leftrightsquigarrow$.

The identity component C_e contains the identity element.

The **order** of $x \in A$ is the smallest positive n such that $x^n \in C_g$, or ∞ if no finite n exists.

Ipo-monoids where $x \leq 0$ or $1 \leq x$ for all $x \in C_e$

Note that $x \leq 0$ or $1 \leq x$ holds in S_m.

Lemma

Let A be an ipo-monoid such that $0 \leq 1$ and ($x \leq 0$ or $1 \leq x$) for all $x \in C_e$.

If $x, y \in C_g$ have order n then $f : C_g \to C_e$ given by $f(x) = x^n$ is injective.

Proof.

Assume $x^n \leq y^n$. We want to show $x \leq y$, then injectivity follows.

Suppose $x \not\leq y$, then $x \cdot {\sim}y \not\leq 0$. Note $x, y \in C_g \implies x \cdot {\sim}y \in C_e$.

From ($z \leq 0$ or $1 \leq z$) for $z \in C_e$ we conclude $1 \leq x \cdot {\sim}y$.

Hence $1 \leq x \cdot {\sim}y \leq x \cdot 1 \cdot {\sim}y \leq x \cdot x \cdot {\sim}y \cdot {\sim}y \leq \cdots \leq x^n \cdot ({\sim}y)^n$.

By the preceding lemma and $x^n \leq y^n$, $x^n \cdot ({\sim}y)^n \leq x^n \cdot {\sim}y^n \leq 0$.

Therefore $x \cdot {\sim}y \leq 0$, a contradiction. □

Structure of finite ipo-monoids with $C_e \cong S_m$

Theorem

Let A be a finite ipo-monoid with $C_e \cong S_m$. Then A is a disjoint union of chains, each with $\leq m$ elements, and $\cdot$ determined by subgroups of $A./\!\leftrightsquigarrow$

Proof (outline).

Let $G = A/\!\leftrightsquigarrow$. The map $f : C_g \to C_e$ is an **order embedding** since $g(x) = x^{n+1}$ is injective on the finite set C_g, hence surjective.

Therefore all connected components are chains with $\leq m$ elements.

For $1 \leq k \leq m$, let $H_k = \{g \in G \mid k \leq |C_g|\}$.

Then $H_m \leq \cdots \leq H_2 \leq H_1 = G$ is a nested sequence of subgroup of G and these subgroups uniquely determine the multiplication on the algebra. □

These algebras can also be constructed using Płonka sums over a linear join-semilattice direct system from ipo-monoids with $C_e = S_2$ and, if $0 = 1$, the least summand has $C_e = S_1$, i.e., it is a group.

And now one more thing! Joint work with Simon Santschi.

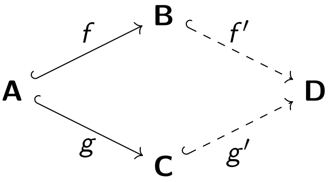
The amalgamation property

A class K of algebras has the **amalgamation property**

if for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathsf{K}$ and embeddings $f\colon \mathbf{A} \to \mathbf{B}$, $g\colon \mathbf{A} \to \mathbf{C}$

there exists $\mathbf{D} \in \mathsf{K}$ and embeddings $f'\colon \mathbf{B} \to \mathbf{D}$, $g'\colon \mathbf{C} \to \mathbf{D}$ such that

$f' \circ f = g' \circ g.$



The pair $\langle f, g\rangle$ is called a **span** and $\langle \mathbf{D}, f', g'\rangle$ is an **amalgam**.

Amalgamation for residuated lattices?

A **residuated lattice** is a rpo-monoid where the partial order is a lattice and $\wedge, \vee$ are included in the signature.

Does **AP** hold for **all residuated lattices**? (**open since** < 2002)

Commutative residuated lattices satisfy $x \cdot y = y \cdot x$

Kowalski, Takamura [2004]: **AP** **holds** for commutative RLs

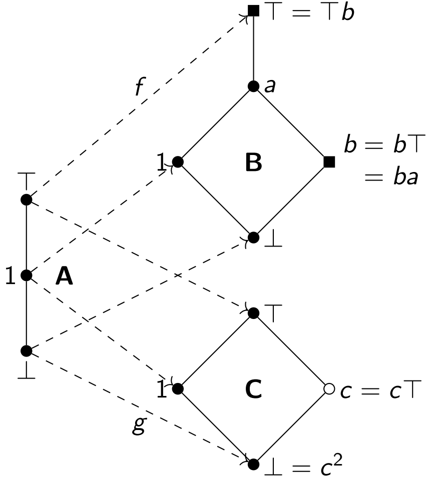
Many other results are know for various subvarieties, e.g.,

Heyting algebras are integral ($x \leq 1$) idempotent ($xx = x$) RLs

Maksimova [1977]: Exactly 8 varieties of Heyting algebras have **AP**

J. and Santschi 2025: **AP fails** for residuated lattices

Theorem: **AP** fails for RL



black = idempotent, round = central

Proof: Straightforward to check $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are RLs and f, g are embeddings.
Assume by contradiction $\exists$ amalgam $\mathbf{D}$.
$1 \vee c = \top$ and $1 \vee b = 1 \vee a = a < \top$
hence $g'(c) \neq f'(a)$ and $g'(c) \neq f'(b)$.
So f', g' are inclusions and $\mathbf{B}, \mathbf{C} \leq \mathbf{D}$
Now, since $c = c\top$ and $\top b = \top$,
in $\mathbf{D}$ we have $cb = c\top b = c\top = c$.
Moreover $\top = 1 \vee c$ and $c^2 = \bot$,
show $c = \top c = \top bc = (1 \vee c)bc$
$= bc \vee cbc = bc \vee c^2 = bc \vee \bot = bc$
(using $\bot \leq c$ implies $\bot = b\bot \leq bc$).
But also $b = b\top = b(1 \vee c) = b \vee bc$
gives $c = bc \leq b \leq a$. Hence
$\top = 1 \vee c \leq a \vee c = a$; contradiction!

Some remarks

The proof on the previous slide also shows that the **AP** already fails for the variety of **distributive residuated lattices**,

as well as for the $\{\backslash, /\}$-free subreducts of residuated lattices, i.e., for **lattice-ordered monoids**.

Also the proof does not depend on meet or on the constant 1 being in the signature, so the following varieties do not have **AP**:

- **residuated lattice-ordered semigroups**,
- **lattice-ordered semigroups**,
- **residuated join-semilattice-ordered semigroups** and
- **join-semilattice-ordered semigroups**.

Similar examples show that **AP** fails in idempotent RLs and in involutive residuated lattices.

Does AP hold for ipo-monoids, rpo-monoids, or integral residuated lattices?

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THANKS!