Residuated lattices do **not** have the amalgamation property

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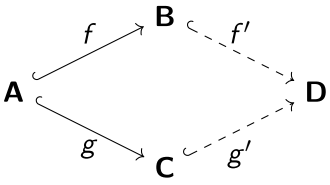
A class K of algebras has the **amalgamation property**

if for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathsf{K}$ and embeddings $f\colon \mathbf{A} \to \mathbf{B}$, $g\colon \mathbf{A} \to \mathbf{C}$

there exists $\mathbf{D} \in \mathsf{K}$ and embeddings $f'\colon \mathbf{B} \to \mathbf{D}$, $g'\colon \mathbf{C} \to \mathbf{D}$ such that

$$f' \circ f = g' \circ g.$$



The pair $\langle f, g\rangle$ is called a **span** and $\langle \mathbf{D}, f', g'\rangle$ is an **amalgam**.

What can we do with the Amalgamation Property?

Bjarni Jónsson

(AMS-MAA meeting in Madison, WI 1968)

Universal relational systems [1956]

E. g. a group **G** is **universal** if

any group of cardinality less or equal

is isomorphic to a subgroup of **G**.

If a class has the amalgamation property

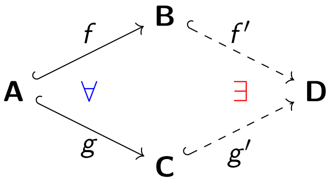
and satisfies a few other mild conditions

then universal members exists in the class

for each uncountable cardinality

The strong amalgamation property

The **amalgamation property** (AP):



such that the diagram commutes.

The **strong amalgamation property** (SAP): in addition to

$f' \circ f = g' \circ g$ we also require $f'[f[A]] = f'[B] \cap g'[C]$

Equivalently: If $\mathbf{A}$ is a subalgebra of $\mathbf{B}, \mathbf{C}$ in K and $\mathbf{A} = \mathbf{B} \cap \mathbf{C}$ then there exists $\mathbf{D} \in \mathsf{K}$ such that $\mathbf{B}, \mathbf{C}$ are subalgebras of $\mathbf{D}$

Some well-known results

The category of sets has the **strong AP**

(Given sets A, B, C with $A = B \cap C$ take $D = B \cup C$.)

Schreier [1927]: Given two groups $\mathbf{B}, \mathbf{C}$ intersecting in a subgroup $\mathbf{A}$, the free product of $\mathbf{B}, \mathbf{C}$ with amalgamated subgroup exists

$\Longrightarrow$ the category of **groups** has the **SAP**

Jónsson [1956] The variety of all **magmas** (sets with a binary operation) has the **SAP**

(Again take $D = B \cup C$ and fill in the remaining values in the operation table of $\mathbf{D}$ arbitrarily.)

The same works for any variety of **all** algebras of a given signature.

More well-known results

Jónsson [1956] The class of **partially ordered sets** has the **SAP**

He also proves there exists a countable universal poset

Jónsson [1956] The variety of all **lattices** has the **SAP**

Kimura [1957] **Semigroups** do **not** have the **AP**

AP also **fails** for the class of **finite** semigroups

Pierce [1968] **AP holds** for **distributive lattices** but **SAP fails**

Pierce [1972] **AP fails** in lattice-ordered groups, **but holds** in abelian lattice-ordered groups

Compendium on amalgamation

Kiss, Márki, Pröhle and Tholen [1983] Categorical algebraic properties. A **compendium on amalgamation**, congruence extension, epimorphisms, residual smallness and injectivity

They summarize some general techniques for establishing these properties

They give a table with **known results for 100 categories**

Day and Jezek [1984] The **only** lattice varieties that satisfy **AP** are the **trivial variety**, the **variety of distributive lattices** and **the variety of all lattices**

Amalgamation for residuated lattices

A **residuated lattice** $(A, \vee, \wedge, \cdot, 1, \backslash, /)$ is an algebra where $(A, \vee, \wedge)$ is a **lattice**, $(A, \cdot, 1)$ is a **monoid** and for all $x, y, z \in A$

$$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z / y$$

Residuated lattices (RLs) generalize many algebras related to **logic**, e. g. **Boolean algebras**, **Heyting algebras**, **MV-algebras**, Hajek's **basic logic algebras**, **linear logic algebras**, ...

Does **AP** hold for **all residuated lattices**? (**open since** < 2002)

Commutative residuated lattices satisfy $x \cdot y = y \cdot x$

Kowalski, Takamura [2004]: **AP holds** for commutative RLs

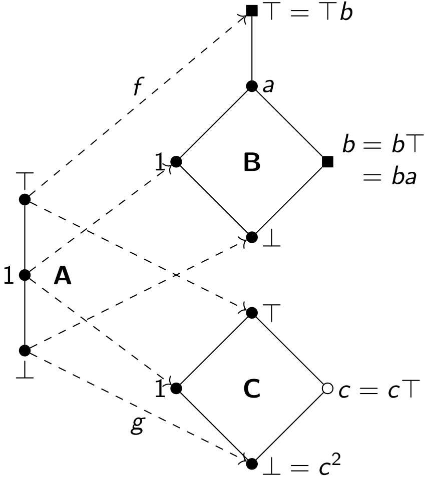
Heyting algebras are integral ($x \leq 1$) idempotent ($xx = x$) RLs

Maksimova [1977]: Exactly 8 varieties of Heyting algebras have **AP**

An interesting table by Metcalfe, Paoli, Tsinakis [2023]

Variety	Label	CEP	CIP	DIP	AP
Residuated lattices	$\mathcal{RL}$	no	yes	?	?
Commutative $\mathcal{RL}$	$\mathcal{CRL}$	yes	yes	yes	yes
Semilinear $\mathcal{RL}$	$\mathcal{SemRL}$	no	no	?	no
Commutative $\mathcal{SemRL}$	$\mathcal{CSemRL}$	yes	no	?	?
GBL-algebras	$\mathcal{GBL}$	no	?	?	no
Commutative $\mathcal{GBL}$	$\mathcal{CGBL}$	yes	?	?	?
Semilinear $\mathcal{GBL}$	$\mathcal{SemGBL}$	no	no	?	no
Commutative $\mathcal{SemGBL}$	$\mathcal{CSemGBL}$	yes	no	yes	yes
GMV-algebras	$\mathcal{GMV}$	no	?	?	no
Commutative $\mathcal{GMV}$	$\mathcal{CGMV}$	yes	no	yes	yes
ℓ-groups	$\mathcal{LG}$	no	?	?	no
Abelian ℓ-groups	$\mathcal{AbLG}$	yes	no	yes	yes
Integral $\mathcal{RL}$	$\mathcal{IRL}$	no	yes	?	?
Commutative $\mathcal{IRL}$	$\mathcal{CIRL}$	yes	yes	yes	yes
Semilinear $\mathcal{IRL}$	$\mathcal{SemIRL}$	no	no	?	?
Commutative $\mathcal{SemIRL}$	$\mathcal{CSemIRL}$	yes	no	?	?
Integral $\mathcal{GBL}$	$\mathcal{IGBL}$	no	?	?	no
Commutative $\mathcal{IGBL}$	$\mathcal{CIGBL}$	yes	?	?	?
Semilinear $\mathcal{IGBL}$	$\mathcal{SemIGBL}$	no	no	?	no
Commutative $\mathcal{SemIGBL}$	$\mathcal{CSemIGBL}$	yes	no	yes	yes
Integral $\mathcal{GMV}$	$\mathcal{IGMV}$	no	?	?	?
Commutative $\mathcal{IGMV}$	$\mathcal{CIGMV}$	yes	no	yes	yes
Negative cones of ℓ-groups	$\mathcal{LG}^-$	no	?	?	no
Abelian $\mathcal{LG}^-$	$\mathcal{AbLG}^-$	yes	no	yes	yes
Brouwerian algebras	$\mathcal{Brw}$	yes	yes	yes	yes
Relative Stone algebras	$\mathcal{RSA}$	yes	yes	yes	yes

Theorem: **AP** fails for RL



black = idempotent, round = central

Proof: Straightforward to check **A**, **B**, **C** are RLs and f, g are embeddings.

Assume by contradiction $\exists$ amalgam **D**.

$1 \vee c = \top$ and $1 \vee b = 1 \vee a = a < \top$

hence $g'(c) \neq f'(a)$ and $g'(c) \neq f'(b)$.

So f', g' are inclusions and **B**, **C** $\leq$ **D**

Now, since $c = c\top$ and $\top b = \top$,

in **D** we have $cb = c\top b = c\top = c$.

Moreover $\top = 1 \vee c$ and $c^2 = \bot$,

show $c = \top c = \top bc = (1 \vee c)bc$

$= bc \vee cbc = bc \vee c^2 = bc \vee \bot = bc$

(using $\bot \leq c$ implies $\bot = b\bot \leq bc$).

But also $b = b\top = b(1 \vee c) = b \vee bc$

gives $c = bc \leq b \leq a$. Hence

$\top = 1 \vee c \leq a \vee c = a$; contradiction!

Some remarks

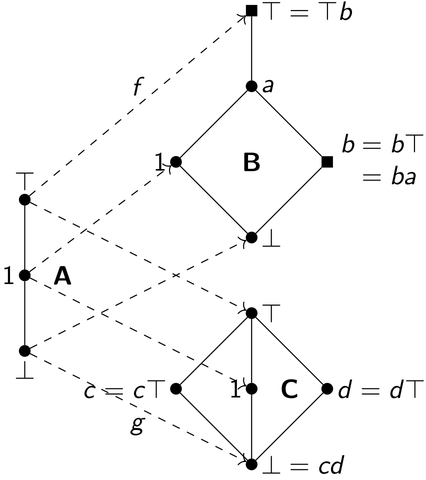
The proof on the previous slide also shows that the **AP** already fails for the variety of **distributive residuated lattices**,

as well as for the $\{\backslash, /\}$-free subreducts of residuated lattices, i.e., for **lattice-ordered monoids**.

Also the proof does not depend on meet or on the constant 1 being in the signature, so the following varieties do not have **AP**:

- **residuated lattice-ordered semigroups**,
- **lattice-ordered semigroups**,
- **residuated join-semilattice-ordered semigroups** and
- **join-semilattice-ordered semigroups**.

Theorem: **AP** fails for idempotent RLs



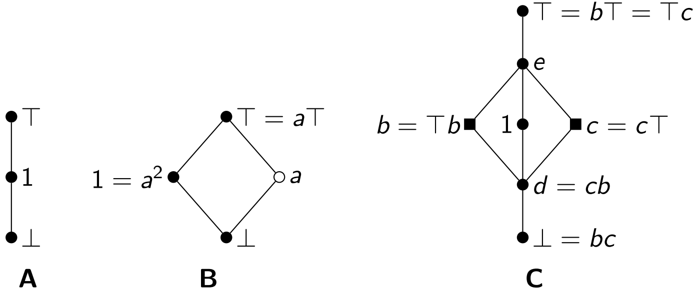
black = idempotent, round = central

Proof: Very similar argument (try it)

For a RL with a new constant 0 define $\sim x = x\backslash 0$, $-x = 0/x$

A RL is **involutive** if $\sim -x = x = -\sim x$

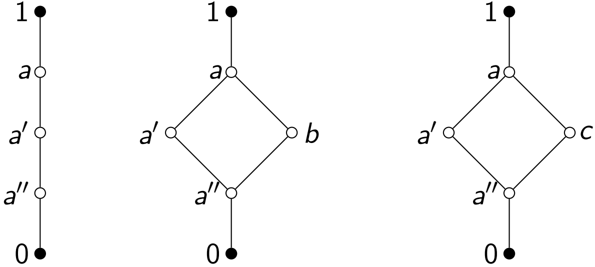
Theorem: **AP** fails for involutive residuated lattices



Note: in all algebras $0 = 1$, black $=$ idempotent, round $=$ central

AP **fails** for distributive residuated lattices

A picture proof:



$\implies b = c$

$$x \cdot y = y \cdot x = \begin{cases} y & \text{if } x = 1 \\ a'' & \text{if } {}^{x \in \{a,b,c\}}_{y \in \{a,a'\}} \\ 0 & \text{otherwise} \end{cases} \qquad b \cdot b = 0 \qquad c \cdot c = a''$$

Proved independently by Galatos 2002, J. 2014, Fussner 2023.

An interesting table by Metcalfe, Paoli, Tsinakis [2023]

				commutative		
Variety	CIP	DIP	AP	CIP	DIP	AP
Residuated **L**attices	yes	?	**no**	yes	yes	yes
Semilinear **RL**	no	?	no	no	?	?
GBL-algebras	?	?	no	?	?	?
Semilinear **GBL**	no	?	no	no	yes	yes
GMV-algebras	?	?	no	no	yes	yes
ℓ-groups	?	?	no	no	yes	yes
Integral **RL**	yes	?	?	yes	yes	yes
Semilinear **IRL**	no	?	?	no	?	?
Integral **GBL**	?	?	no	?	?	?
Semilinear **IGBL**	no	?	no	no	yes	yes
Integral **GMV**	?	?	?	no	yes	yes
Negative cones of ℓ-groups	?	?	no	no	yes	yes
Brouwerian algebras	—	—	—	yes	yes	yes
Relative stone algebras	—	—	—	yes	yes	yes

An interesting table by Metcalfe, Paoli, Tsinakis [2023]

				commutative		
Variety			AP			AP
Residuated **L**attices			**no**			yes
Semilinear **RL**			no			?
GBL-algebras			no			?
Semilinear **GBL**			no			yes
GMV-algebras			no			yes
ℓ-groups			no			yes
Integral **RL**			?			yes
Semilinear **IRL**			?			?
Integral **GBL**			no			?
Semilinear **IGBL**			no			yes
Integral **GMV**			?			yes
Negative cones of ℓ-groups			no			yes
Brouwerian algebras			—			yes
Relative stone algebras			—			yes

Many other results are known...

For example:

Fusser, Metcalfe and Santschi [2023] showed that there are **exactly 60 varieties of commutative idempotent semilinear residuated lattices have the amalgamation property**.

Giustarini and Ugolini [2024] proved that **semilinear commutative (integral) residuated lattices** and their pointed versions **do not** have the amalgamation property.

...

				commutative		
Variety			AP			AP
Residuated **L**attices			**no**			yes
Semilinear **RL**			no			**no**
Distributive **RL**			**no**			**no**
Integral **RL**			?			yes
Idempotent **RL**			**no**			yes
Involutive **RL**			**no**			yes
DIRL			**no**			**no**
DIdRL			?			?
DInRL			**no**			**no**
GBL-algebras			no			?
GMV-algebras			no			yes
ℓ-groups			no			yes
Semilinear **IRL**			?			**no**
Integral **GBL**			no			?
Integral **GMV**			?			yes
Negative cones of ℓ-groups			no			yes

Galatos 2002, Fussner 2023, Giustarini, Ugolini 2024, J., Santschi 2025

How we searched for failures of the **AP**

To **disprove AP** or **SAP**, we wish to search for 3 **small** models A, B, C in $\mathcal{K}$ such that A is a **submodel** of both B and C.

We used the **Mace4 model finder** from **Bill McCune [2009]** to enumerate nonisomorphic models $A_1, A_2, \ldots$ in a **finitely axiomatized** first-order theory Σ.

For each A_i we construct the **positive diagram** Δ_i^+ and use **Mace4** again to find all **nonisomorphic** models $B_1, B_2, \ldots$ of $\Delta_i^+ \cup \Sigma \cup \{\neg(c_a = c_b) : a \neq b \in A_i\}$ with **slightly more** elements than A_i.

Note that **by construction**, each B_j has A_i as submodel.

Checking if the **AP** fails

Iterate over **distinct** pairs of models B_j, B_k and construct the theory Γ that extends Σ with the **positive diagrams of these two models**, using only **one set of constants** for the overlapping submodel A_i.

Add formulas to Γ that ensure all constants of B_j are **distinct**, and same for B_k.

Use **Mace4** to check for a **limited** time whether Γ is satisfiable in some **small** model.

If not, use the **Prover9 automated theorem prover** (McCune [2009]) to search for a proof that Γ is **inconsistent**. If **yes**, then a **failure of AP** has been found.

To check if **SAP fails**, add formulas that ensure constants of **each pair** of models **cannot** be identified, and **also iterate** over pairs B_j, B_j.

Some open problems

Does the variety of **integral residuated lattices** have the amalgamation property?

Find the **amalgamation base** of residuated lattices (all **A** such that any span using **A** can be amalgamated).

References

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Thanks!