

From groups to residuated semigroups and other partially ordered algebras

Peter Jipsen

joint work with Erkki Lehtonen and Reinhard Pöschel

Chapman University, Khalifa University, University of Dresden

AGK Seminar, Universität Dresden, July 3, 2025

Outline

- Groups (from a slightly different perspective)
- Partially ordered groups
- Residuated partially ordered semigroups (= r-semigroups)
- Examples: relation algebras
- Płonka sums of r-semigroups (Bonzio, Gil-Férez, Prenosil, Sugimoto)
- Amalgamation for residuated lattices? (joint with S. Santschi)
- Partially ordered algebras
- Clones and preclones (joint with E. Lehtonen and R. Pöschel)
- \pm -preclones and S -preclones
- A Pol-Inv Galois connection for S -preclones
- The Post lattice and the lattice of \pm -preclones
- Maximal and minimal Boolean \pm -preclones

Semigroups with division

A **semigroup with division** is an algebra $\mathbf{A} = (A, \cdot, /, \backslash)$ such that

① \cdot is associative and

$$\textcircled{2} \quad xy = z \iff x = z/y \iff y = x \backslash z.$$

So every linear equation can be solved for each variable.

(2) on its own defines **quasigroups**.

Has anyone heard of **associative quasigroups**?

Turns out they satisfy $x/x = y \backslash y$ hence they are **groups**.

$$x^{-1} = (x/x)/x \text{ and } x \backslash y = y/x \text{ (good exercise).}$$

Partially ordered groups

A **partially ordered group** is a structure $\mathbf{A} = (A, \leq, \cdot, {}^{-1}, 1)$ such that

① (A, \leq) is a poset

② $(A, \cdot, 1)$ is a monoid and

③ $xy \leq z \iff x \leq zy^{-1} \iff y \leq x^{-1}z.$

$\implies \cdot$ is order-preserving in both arguments

${}^{-1}$ is order-reversing

Some examples: $(\mathbb{R}, \leq, +, -, 0)$, $(\mathbb{Q}^+, \leq, \cdot, {}^{-1}, 1)$

all groups (where \leq is $=$)

Residuated (partially ordered) semigroups

A **residuated semigroup** is a structure $\mathbf{A} = (A, \leq, \cdot, /, \backslash)$ such that

- ① (A, \leq) is a poset
- ② \cdot is associative and
- ③ $xy \leq z \iff x \leq z/y \iff y \leq x \backslash z$ (**residuation**).

A **residuated monoid** is a residuated semigroup expanded with a constant operation 1 such that $1x = x1 = x$.

$\implies \cdot$ is order-preserving in both arguments

$/$ is order-preserving in the first and order-reversing in the second argument

Some examples: po-groups, $\text{Rel}(X) = (P(X \times X), \subseteq, \circ, /, \backslash)$

If \leq is the $=$ relation, we get **groups** (see semigroups with division).

Balanced residuated semigroups

A residuated semigroup is **balanced** if $x/x = x \backslash x$

idempotent if $x^2 = x$ (where $x^2 = x \cdot x$)

integral if $x \leq 1$ (i.e. 1 is the top element)

Examples: (1) every commutative residuated lattice is balanced,

(2) every Boolean algebra is idempotent and integral.

A **residuated lattice** is a residuated monoid that is a lattice (has \vee, \wedge).

The structure of residuated semigroups

Want to **decompose** (certain) residuated semigroups into **simpler fibers**.

The fibers are residuated semigroups with a **unique positive idempotent**.

Reconstruction uses a **Plonka sum over a semilattice of two families of maps**.

Extends structure of **even/odd involutive FL_e -chains** [Jenei 2022],

finite commutative idempotent involutive residuated lattices, the fibers are **Boolean algebras** [Jipsen, Tuyt, Valota 2021], and

locally integral involutive po-semigroups, where the fibers are **integral involutive residuated monoids** [Gil-Férez, Jipsen, Sugimoto 2024].

How many small residuated monoids are there?

Cardinality $n =$	1	2	3	4	5	6	7	8	9
Residuated monoids (RM)	1	2	5	28	186	1795	18554	295292	
Residuated lattices	1	1	3	20	149	1488			
Balanced RM	1	2	5	26	184	1461			
Unique pos. idem. RM	1	2	4	16	72	516	3348		
Integral RM	1	1	2	9	49	365			
Idempotent RM	1	1	2	8	32	148	785		
Comm. idempotent RM	1	1	2	6	20	78	351	1798	26
Idempotent integral RM	1	1	1	2	3	5	8	15	

A residuated semigroup is **commutative** if $xy = yx$.

Can we find good structure theory for some of these classes?

Examples of good structure theory

Cardinality $n =$	1	2	3	4	5	6	7	8	9	10	11	12
Boolean algebras	1	1	0	1	0	0	0	1	0	0	0	0
Fields	1	1	1	1	1	0	1	1	1	0	1	0
Abelian groups	1	1	1	2	1	1	1	3	2	1	1	2
MV-algebras	1	1	1	2	1	2	1	3	2	2	1	4
Gödel algebras	1	1	1	2	2	3	3	5	6	8	8	12
Distr. lattices	1	1	1	2	3	5	8	15	26	47	82	151
BL-algebras	1	1	2	5	9	20	38	81	160	326	643	1314

Note that all these examples are commutative.

Cardinality $n =$	1	2	3	4	5	6	7	8	9
Partially ordered monoids	1	6	109	5849					
Residuated monoids (RM)	1	2	5	28	186	1795			
Residuated lattices	1	1	3	20	149	1488	18554	295292	
Idempotent integral RM	1	1	1	2	3	5	8	15	26

Small residuated monoids

There is a unique residuated monoid (RM) of cardinality 1.

There are 2 RMs of cardinality 2: the BA **2** and the 2-elt group \mathbb{Z}_2

There are 5 RMs of cardinality 3: **G**₃, **L**₃, **S**₃, \mathbb{Z}_3 and **2+1**

$$\begin{array}{c} 1 \\ \bullet \\ \circ u \\ \bullet \\ 0 \end{array} \quad 0x = x0 = 0, 1x = x1 = x, u^2 = 0$$

There are 28 RMs of cardinality 4: 15 are linearly ordered, 5 are based on **2×2**, 3 are based on **3+1**, 2 are based on **2+2**, **2+1+1**, \mathbb{Z}_4 , $\mathbb{Z}_2 \times \mathbb{Z}_2$

Later: how to build bigger residuated semigroups from these pieces.

A list of small residuated lattices

RESIDUATED LATTICES OF SIZE UP TO 6

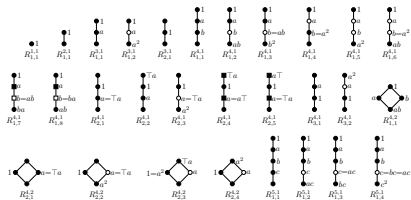
NICK GALATOS AND PETER JIPSEN

There are $1 + 1 + 3 + 20 + 149 + 1488 = 1662$ residuated lattices with ≤ 6 elements. In the list below, each algebra is named $R_{ij}^{m,n}$ where m is the cardinality and n enumerates nonisomorphic lattices of size m , in order of decreasing height. The depth of the identity element 1 is given by i , and j enumerates nonisomorphic algebras. For lattices of the same height distributive lattices appear before modular lattices, followed by nonmodular lattices, and selfdual lattices appear before nonselfdual lattices. Algebras with more central elements (round circles) are listed earlier, hence commutative residuated lattices precede noncommutative ones. Finally, algebras are listed in decreasing order of number of idempotents (black nodes).

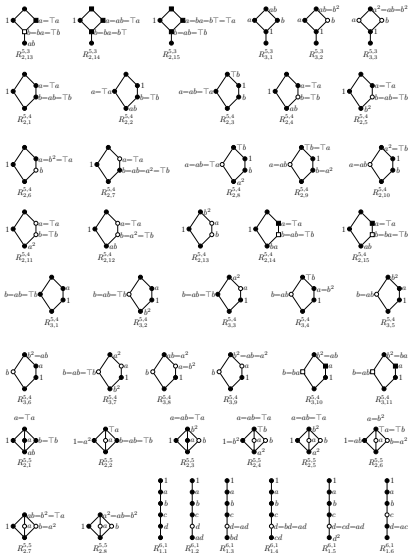
The monoid operation is indicated by labels. If a nonbivisible product xy is not listed, then it can be deduced from the given information: either it follows from idempotence ($x^2 = x$) indicated by a black node or from commutativity or there are products $uv = w_1$ such that $u \leq x \leq w$ and $v \leq y \leq z$ (possibly $w = \perp$ or $w = \top$).

If you have comments or notice any issues in this list, please email jipsen.AT.chapman.edu.

- = central idempotent
- = central nonidempotent
- = noncentral idempotent
- = noncentral nonidempotent



Date: April 16, 2017.

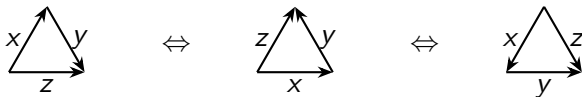


Relation algebras

A **relation algebra** $\mathbf{A} = (A, \sqcup, {}^c, ;, 1, {}^{-1})$ is a monoid $(A, ;, 1)$

and a Boolean algebra $(A, \sqcup, {}^c)$ with operation ${}^{-1}$ that satisfies

$$x;y \sqcap z = \perp \Leftrightarrow z;y^{-1} \sqcap x = \perp \Leftrightarrow x^{-1};z \sqcap y = \perp$$



; has priority over the meet $x \sqcap y = (x^c \sqcup y^c)^c$, and $\perp = 1 \sqcap 1^c$

Did you see the relation algebras on the **previous slide**?

Term-reductions of relation algebras are residuated lattices: $x/y = (x^c; y^{-1})^c$

Representable relation algebras RRA

The **full algebra of binary relations on** a set X is

$$\mathbf{Rel}(X) = (\mathcal{P}(X^2), \cup, ^c, ;, id_X, ^{-1}) \text{ where } R^c = X^2 \setminus R$$

composition $R ; S = \{(x, y) \mid \exists z, (x, z) \in R \text{ and } (z, y) \in S\}$

converse $R^{-1} = \{(x, y) \mid (y, x) \in R\}$

Proposition: $\mathbf{Rel}(X)$ is a relation algebra

RRA = representable relation algebras = $\mathbb{SP}\{\mathbf{Rel}(X) \mid X \text{ is a set}\}$

For more details see the books by Givant [2017] and Maddux [2006]

A database of finite integral relation algebras up to 5 atoms

An **atom** in a BA is a smallest element $\neq \perp$.

An atom a in a RA is **symmetric** if $a = a^{-1}$.

Let a, b, c, d be symmetric atoms ($x^{-1} = x$) and r, s nonsymmetric

The number of RAs up to isomorphism is given below:

Cardinality	2	4	8	8	16	16	32	32	32
Atoms	1	1a	1rr ⁻¹	1ab	1arr ⁻¹	1abc	1rr ⁻¹ ss ⁻¹	1abrr ⁻¹	1abcd
Total	1	2	3	7	37	65	83	1316	3013
Representable	1	2	3	7	26	45	39	298	480
Nonrepresent.	0	0	0	0	11	20	29	783	2048
Unknown	0	0	0	0	0	0	15	235	485

For the list of 83 there are 15 RAs not known to be (non)representable:
30,31,32,40,44,45,54,56,59,60,61,63,65,69,79 (see [Maddux 2006])

Łonka sums of semigroups

An interesting way to build an algebra from a **union of algebras**:

Let (I, \vee) be a semilattice and $\{\varphi_{ij} : \mathbf{A}_i \rightarrow \mathbf{A}_j \mid i \leq j \in I\}$ a family of homomorphisms of **disjoint** semigroups \mathbf{A}_i such that $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$.

Then $\mathbf{A} = (\bigcup_{i \in I} A_i, \cdot)$ is a semigroup if for $x \in A_i, y \in A_j$ and $k = i \vee j$

we define $x \cdot y = \varphi_{ik}(x) \cdot_k \varphi_{jk}(y)$.

\mathbf{A} is the **Łonka sum** of the family of homomorphisms.

If the \mathbf{A}_i are groups, the Łonka sum produces **Clifford semigroups**.

Łonka sums of residuated semigroups

Can we construct **unions of po-algebras** in a similar way?

Theorem

Let (I, \vee) be a semilattice and $\{\varphi_{ij}, \psi_{ij} : \mathbf{A}_i \rightarrow \mathbf{A}_j \mid i \leq j \in I\}$ two families of maps of **disjoint** r -semigroups \mathbf{A}_i such that for all $i \leq j \leq k$

- $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$, $\psi_{jk} \circ \psi_{ij} = \psi_{ik}$, $\varphi_{ij}(xy) = \varphi_{ij}(x)\varphi_{ij}(y)$,
- $\psi_{ij}(x \setminus y) = \varphi_{ij}(x) \setminus \psi_{ij}(y)$, $\psi_{ij}(x / y) = \psi_{ij}(x) / \varphi_{ij}(y)$,
- if $i \leq j, k$ and $\ell = j \vee k$, then $\varphi_{j\ell} \circ \psi_{ij} \leq_\ell \psi_{k\ell} \circ \varphi_{ik}$ pointwise, and
- if $i \leq j$ and $a, b \in A_i$ and $\varphi_{ij}(a) \leq_j \psi_{ij}(b)$, then $a <_i b$.

Then $\mathbf{A} = (\bigcup_{i \in I} A_i, \cdot, /, \setminus)$ is an r -semigroup if for $k = i \vee j$ we define

$$x \cdot y = \varphi_{ik}(x) \cdot_k \varphi_{jk}(y), \quad x / y = \psi_{ik}(x) /_k \varphi_{jk}(y),$$

$$x \setminus y = \varphi_{ik}(x) \setminus_k \psi_{jk}(y) \quad \text{and} \quad x \leq y \iff \varphi_{ik}(x) \leq_k \psi_{jk}(y).$$

\mathbf{A} is the **Łonka sum** of the family of **meta**morphisms (φ, ψ) .

All steady residuated semigroups are Płonka sums

In a balanced $(x/x = x \backslash x)$ r-semigroup, define the term $1_x = x/x$.

A balanced r-semigroup **A** is **steady** if it satisfies $1_{xy} = 1_x 1_y = 1_{x/y}$

Theorem

Let $I = \{p \in A \mid x \leq px, x \leq xp, pp = p\}$, define the fibers

$A_p = \{x \in A \mid 1_x = p\}$, and maps $\varphi_{pq}(x) = xq$, $\psi_{pq}(x) = x/q$.

Then the $\mathbf{A}_p = (A_p, \leq_p, \cdot_p, /_p, \backslash_p, p)$ are residuated monoids with the induced operations, unique positive idempotents, the conditions of the previous theorem hold and the Płonka sum of these fibers reconstructs **A**.

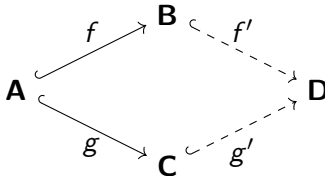
The amalgamation property

A class K of algebras has the **amalgamation property**

if for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in K$ and embeddings $f: \mathbf{A} \rightarrow \mathbf{B}$, $g: \mathbf{A} \rightarrow \mathbf{C}$

there exists $\mathbf{D} \in K$ and embeddings $f': \mathbf{B} \rightarrow \mathbf{D}$, $g': \mathbf{C} \rightarrow \mathbf{D}$ such that

$$f' \circ f = g' \circ g.$$



The pair $\langle f, g \rangle$ is called a **span** and $\langle \mathbf{D}, f', g' \rangle$ is an **amalgam**.

Amalgamation for residuated lattices?

Does **AP** hold for **all residuated lattices**? (**open since** < 2002)

Commutative residuated lattices satisfy $x \cdot y = y \cdot x$

Kowalski, Takamura [2004]: **AP holds** for commutative RLs

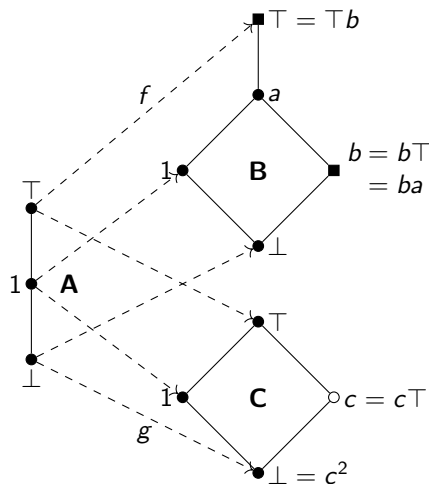
Many other results are known for various subvarieties, e.g.,

Heyting algebras are integral ($x \leq 1$) idempotent ($xx = x$) RLs

Maksimova [1977]: Exactly 8 varieties of Heyting algebras have **AP**

J. and Santschi 2025: **AP** fails for residuated lattices

Theorem: **AP** fails for RL



black = idempotent, round = central

Proof: Straightforward to check **A**, **B**, **C** are RLs and f, g are embeddings.

Assume by contradiction \exists amalgam **D**.

$1 \vee c = T$ and $1 \vee b = 1 \vee a = a < T$ hence $g'(c) \neq f'(a)$ and $g'(c) \neq f'(b)$.

So f', g' are inclusions and **B**, **C** \leq **D**

Now, since $c = cT$ and $Tb = T$, in **D** we have $cb = cTb = cT = c$.

Moreover $T = 1 \vee c$ and $c^2 = \perp$, show $c = Tc = Tbc = (1 \vee c)bc = bc \vee cbc = bc \vee c^2 = bc \vee \perp = bc$ (using $\perp \leq c$ implies $\perp = b\perp \leq bc$).

But also $b = bT = b(1 \vee c) = b \vee bc$ gives $c = bc \leq b \leq a$. Hence

$T = 1 \vee c \leq a \vee c = a$; contradiction!

Some remarks

The proof on the previous slide also shows that the **AP** already fails for the variety of **distributive residuated lattices**,

as well as for the $\{\backslash, /\}$ -free subreducts of residuated lattices, i.e., for **lattice-ordered monoids**.

Also the proof does not depend on meet or on the constant 1 being in the signature, so the following varieties do not have **AP**:

- **residuated lattice-ordered semigroups**,
- **lattice-ordered semigroups**,
- **residuated join-semilattice-ordered semigroups** and
- **join-semilattice-ordered semigroups**.

Similar examples show that **AP** fails in idempotent RLs and in involutive FL-algebras.

Partially ordered algebras

Unsorted universal algebra has been developed over the last century

Central concepts: signature, algebras, homomorphisms, congruences, subalgebras, products, HSP, varieties, free algebras, clones, ...

Every algebra $\mathbf{A} = (A, (f^{\mathbf{A}} \mid f \in \mathcal{F}))$ has an underlying **set** A as its universe

A **partially ordered algebra** $\mathbf{A} = (A, \leq^{\mathbf{A}}, (f^{\mathbf{A}} \mid f \in \mathcal{F}))$ has an underlying **poset** $A = (A, \leq^{\mathbf{A}})$, as its universe; the **dual poset** A^{∂} is $(A, \geq^{\mathbf{A}})$

Each **fundamental operation symbol** $f \in \mathcal{F}$ corresponds to an operation $f^{\mathbf{A}}$ of \mathbf{A} that is order-preserving **or order-reversing** in each argument

This information is part of the **signature** f^{λ} where $\lambda \in \bigcup_{n \geq 1} \{+, -\}^n$

E.g. $f^{(+,-)}$ means f is binary and $f^{\mathbf{A}}: A \times A^{\partial} \rightarrow A$ is order-preserving

Partially ordered algebras

In standard universal algebra the base category is the category of **sets**

For **partially ordered algebras (po-algebras)** the base category is

Pos = the category of posets with order-preserving maps as morphisms

But term-operations on **A** are not necessarily morphisms in **Pos**

For $f^{(+,-)}$, $f^{\mathbf{A}}(x, x)$ may **not** be order-preserving or order-reversing

Varieties of algebras with order-preserving operations have been studied by [Bloom 1976], [Bloom and Wright 1983], [Kurz and Velebil 2017], ...

However for algebraic logic, **negation** and **residuation** are important operations, and they are **not** order-preserving

The study of (nonorder-preserving) po-algebras is due to [Pigozzi 2004]

Clones of algebras

On the second slide we saw semigroups with division “are” groups.

Actually they are **term-equivalent**, i.e., they have the same clone of term-operations.

The **term clone** of an algebra **A** is the set of term-operations built from the basic operations listed in the algebra.

In algebra we study clones of algebras to avoid choosing specific basic operations that generate the clone.

E.g., Boolean algebras and Boolean rings have the same clone

Definition of clone

For a set A , let $\text{Op}(A) = \bigcup_{n \geq 1} A^{A^n}$ = the set of all finitary operations on A

A set $F \subseteq \text{Op}(A)$ is a **clone** on A if

F contains the **identity operation** id_A and

F is closed under $\zeta, \tau, \nabla, \Delta, \circ$ where

- ζ **cycles** arguments: $(\zeta f)(x_1, \dots, x_n) = f(x_2, \dots, x_n, x_1)$
- τ **permutes** first arguments: $(\tau f)(x_1, \dots, x_n) = f(x_2, x_1, x_3, \dots, x_n)$
- ∇ **adds a fictitious** argument: $(\nabla f)(x_1, \dots, x_n) = f(x_2, \dots, x_n)$
- Δ **identifies first 2** arguments: $(\Delta f)(x_1, \dots, x_n) = f(x_1, x_1, x_2, \dots, x_n)$
- \circ **composes** f, g : $(f \circ g)(x_1, \dots, x_{m+n}) = f(g(x_1, \dots, x_m), x_{m+1}, \dots, x_{m+n})$

The lattice of all clones on a set

The collection \mathcal{L}_A of all clones on a set A is closed under intersections.

So it forms a **complete lattice** under subset inclusion.

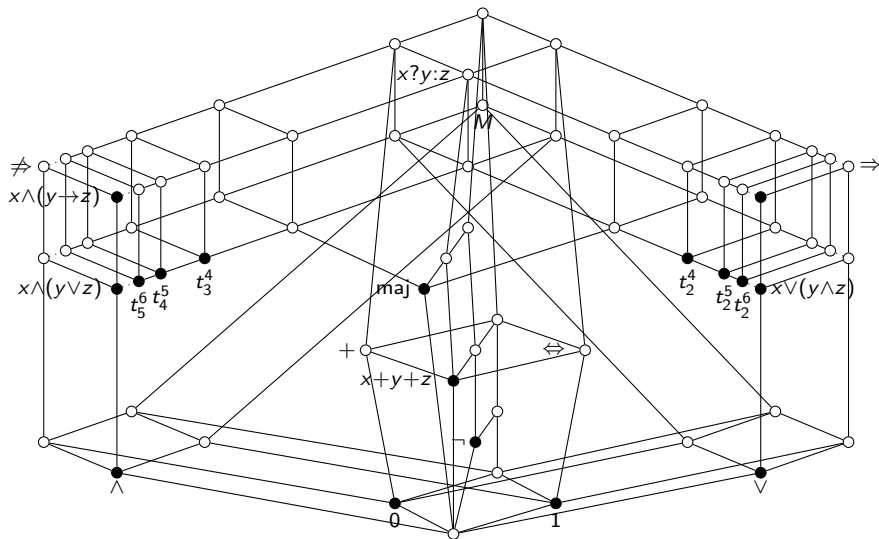
$\langle X \rangle = \bigcap \{F \mid X \subseteq F \text{ a clone}\} = \text{clone generated by } F.$

$$F \vee G = \langle F \cup G \rangle$$

The top is $\text{Op}(A)$ and the bottom is $\{p_i^n \mid i \leq n \in \mathbb{Z}^+\}$

where $p_i^n(x_1, \dots, x_n) = x_i$ is the i^{th} **projection**.

The Post lattice of all clones on a 2-element set



$$t_k^n(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } |\{i \mid x_i = 1\}| \geq k \\ 0 & \text{otherwise.} \end{cases}$$

Operations with signa from a monoid S

The set of term-operations on an algebra form a **clone**

What is a good notion of **clone for partially ordered algebras**?

For a set A , let $\text{Op}(A) = \bigcup_{n \geq 1} A^{A^n}$ = the set of all finitary operations on A

For a (fixed) **finite monoid** (S, \cdot, e) , let ${}^S\text{Op}(A) = \bigcup_{n \geq 1} A^{A^n} \times S^n$

A pair $(f, \lambda) \in {}^S\text{Op}(A)$ is an **S -operation** with **signum** λ , also written f^λ

E.g. $S = \{+, -\}$ with $e = +$ and $- \cdot - = +$

$f^{(+,-)}$ is a binary S -operation with $+$ sign and $-$ sign as arguments

S-preclones

A set $F \subseteq {}^S\text{Op}(A)$ is an **S-preclone** if

- F contains the **identity S-operation** $(\text{id}_A, (e))$ and
- F is closed under $\zeta, \tau, \nabla^s, \Delta, \circ$ where $s \in S$ and
 - ζ **cycles** arguments:
 $(\zeta f^\lambda)(x_1, \dots, x_n) = f(x_2, \dots, x_n, x_1)$ has signum $(\lambda_n, \lambda_1, \dots, \lambda_{n-1})$
 - τ **permutes** the first two arguments:
 $(\tau f^\lambda)(x_1, \dots, x_n) = f(x_2, x_1, x_3, \dots, x_n)$ has signum $(\lambda_2, \lambda_1, \lambda_3, \dots, \lambda_n)$
 - ∇^s **adds a fictitious** argument with signum s at first place:
 $(\nabla^s f^\lambda)(x_1, \dots, x_{n+1}) = f(x_2, \dots, x_{n+1})$ has signum $(s, \lambda_1, \dots, \lambda_n)$
 - Δ **identifies first two** arguments **if they have equal signs**:
 $(\Delta f^\lambda)(x_1, \dots, x_{n-1}) = f(x_1, x_1, x_2, \dots, x_{n-1})$ has signum $(\lambda_2, \dots, \lambda_n)$ if $\lambda_1 = \lambda_2$, otherwise $\Delta f^\lambda = f^\lambda$
 - \circ **composes** f^λ with g^μ and uses the monoid to get the signum:
 $(f^\lambda \circ g^\mu)(x_1, \dots, x_{m+n-1}) = f(g(x_1, \dots, x_m), x_{m+1}, \dots, x_{m+n-1})$ with **signum** $(\mu_1 \lambda_1, \dots, \mu_m \lambda_1, \lambda_2, \dots, \lambda_n)$

Examples of S -preclones $F \subseteq {}^S\text{Op}(A)$

If $S = \{e\}$ then F is an S -preclone **iff** $\{f \mid f^\lambda \in F\}$ is a clone

If $S = \{+, -\}$ with $e = +$, $- \cdot - = +$ and if F is all operations that are order-preserving in arguments with $+$ sign and order-reversing otherwise, then F is an S -preclone, also called a **\pm -preclone**

${}^S\text{Op}(A)$ is the **largest** S -preclone on a set A

Let $p_i^n(x_1, \dots, x_n) = x_i$. Then $\{(p_i^n, \lambda) \mid \lambda_i = e, 1 \leq i \leq n \in \mathbb{Z}^+\}$ is the **smallest** S -preclone of **trivial S -operations** or **projections**

${}^S\langle F \rangle$ is the S -preclone **generated by** F (= least one containing F)

Relations, polymorphisms and invariant relations

For $f : A^n \rightarrow A$, $\rho \subseteq A^m$ and $\mathbf{a}_1, \dots, \mathbf{a}_n \in \rho$, consider the column vector

$$f(\mathbf{a}_1, \dots, \mathbf{a}_n) = (f(a_{11}, a_{21}, \dots, a_{n1}), \dots, f(a_{m1}, a_{m2}, \dots, a_{mn}))^T.$$

f **preserves** ρ , written $f \triangleright \rho$, if $\mathbf{a}_1, \dots, \mathbf{a}_n \in \rho \implies f(\mathbf{a}_1, \dots, \mathbf{a}_n) \in \rho$.

In this case f is a **polymorphism** of ρ , and ρ is an **invariant relation** of f .

For $F \subseteq \text{Op}(A)$ and $R \subseteq \text{Rel}(A) = \bigcup_{m \geq 1} \mathcal{P}(A^m)$, define

$\text{Pol}(R) = \{f \in \text{Op}(A) \mid \forall \rho \in R, f \triangleright \rho\}$ = all **polymorphisms of R** and

$\text{Inv}(F) = \{\rho \in \text{Rel}(A) \mid \forall f \in F, f \triangleright \rho\}$ = all **invariant relations of F**

E.g. $\text{Pol}(\{\leq\})$ = all **order-preserving operations** on a poset (A, \leq)

Relational clones and the Pol–Inv Galois connection

A set $R \subseteq \text{Rel}(A)$ is a **relational clone** if

- R contains the **identity relation** $\delta_A = \{(x, x) \mid x \in A\}$ and
- R is **closed under** $\zeta, \tau, \text{pr}, \times, \wedge$ where for $\rho, \rho' \in R$

$$\zeta\rho = \{(a_2, a_3, \dots, a_m, a_1) \mid (a_1, a_2, \dots, a_m) \in \rho\} \text{ **cyclic shift**}$$

$$\tau\rho = \{(a_2, a_1, a_3, \dots, a_m) \mid (a_1, a_2, \dots, a_m) \in \rho\} \text{ **transposition**}$$

$$\text{pr}\rho = \{(a_2, \dots, a_m) \mid (a_1, a_2, \dots, a_m) \in \rho\} \text{ **deletion of 1st row**}$$

$$\rho \times \rho' = \{(a_1, \dots, a_m, b_1, \dots, b_{m'}) \mid (a_1, \dots, a_m) \in \rho, (b_1, \dots, b_{m'}) \in \rho'\}$$

$$\rho \wedge \rho' = \rho \cap \rho' \quad \text{ **intersection** } \qquad \text{ **Cartesian product** }$$

$[R]$ is the relational clone **generated by** $R \subseteq \text{Rel}(A)$ (least one $\supseteq R$)

Theorem (Pol–Inv Galois connection for clones)

$\text{Pol}(R)$ is a **clone** and $\text{Inv}(F)$ is a **relational clone**.

If A is **finite**, $\langle F \rangle = \text{Pol}(\text{Inv}(F))$ and $[R] = \text{Inv}(\text{Pol}(R))$.

S-relations, S-polymorphisms and S-invariant relations

An m -ary **S-relation** is of the form $\rho = (\rho_s)_{s \in S}$ where $\rho_s \subseteq A^m$

The set of **all S-relations** is ${}^S\text{Rel}(A) = \bigcup_{m \geq 1} (P(A^m))^S$

Define $f^\lambda \triangleright \rho$ if $\forall s \in S \ (\mathbf{a}_1 \in \rho_{\lambda_1 s}, \dots, \mathbf{a}_n \in \rho_{\lambda_n s} \implies f(\mathbf{a}_1, \dots, \mathbf{a}_n) \in \rho_s)$

Abbreviated as: $\forall s \in S, f(\rho_{\lambda_1 s}, \dots, \rho_{\lambda_n s}) \subseteq \rho_s$

E.g. $S = \{+, -\}$: $f^\lambda \triangleright \rho$ if $f(\rho_{\lambda_1}, \dots, \rho_{\lambda_n}) \subseteq \rho_+$ and $f(\rho_{\lambda_1 -}, \dots, \rho_{\lambda_n -}) \subseteq \rho_-$

$f^\lambda \triangleright (\leq, \geq)$ **iff** f is order-preserving in $+$ args and order-reversing in $-$ args

For $F \subseteq {}^S\text{Op}(A)$ and $R \subseteq {}^S\text{Rel}(A)$ define

${}^S\text{Pol}(R) = \{f^\lambda \in {}^S\text{Op}(A) \mid \forall \rho \in R, f^\lambda \triangleright \rho\}$ = all **S-polymorphisms of R**

${}^S\text{Inv}(F) = \{\rho \in {}^S\text{Rel}(A) \mid \forall f^\lambda \in F, f^\lambda \triangleright \rho\}$ = all **S-invariant relations of F**

S-relational clones

A set $R \subseteq {}^S\text{Rel}(A)$ is a **S-relational clone** if

- R contains the S -relation $(\delta_A)_{s \in S} = (\delta_A, \dots, \delta_A)$ and
- R is **closed under** $\zeta, \tau, \text{pr}, \times, \wedge, \wedge_{St}, \mu_t$ where for $\rho, \rho' \in R$ and $t \in S$

$$\zeta(\rho) = (\zeta \rho_s)_{s \in S} \quad \text{cyclic shift}$$

$$\tau(\rho) = (\tau \rho_s)_{s \in S} \quad \text{transposition}$$

$$\text{pr}(\rho) = (\text{pr } \rho_s)_{s \in S} \quad \text{deletion of 1st row}$$

$$\rho \times \rho' = (\rho_s \times \rho'_s)_{s \in S} \quad \text{Cartesian product}$$

$$\rho \wedge_{St} \rho' = (\rho_s \cap \rho'_s \text{ if } s \in St \text{ else } \rho_s)_{s \in S} \quad \text{St-intersection, } St = \{s \cdot t \mid s \in S\}$$

$$\mu_t(\rho) = (\rho_{st})_{s \in S} \quad \text{index translation by } t$$

${}^S[R]$ is the S -relational clone **generated by** $R \subseteq {}^S\text{Rel}(A)$ (least one $\supseteq R$)

Lemma (For $F \subseteq {}^S\text{Op}(A)$ and $R \subseteq {}^S\text{Rel}(A)$)

${}^S\text{Pol}(R)$ is an **S-preclone** and ${}^S\text{Inv}(F)$ is an **S-relational clone**.

The ${}^S\text{Pol}$ – ${}^S\text{Inv}$ Galois connection

Theorem

Let A be a finite set, $F \subseteq {}^S\text{Op}(A)$, and $R \subseteq {}^S\text{Rel}(A)$.

Then ${}^S\langle F \rangle = {}^S\text{Pol}({}^S\text{Inv}(F))$ and ${}^S[R] = {}^S\text{Inv}({}^S\text{Pol}(R))$.

The set ${}^S\mathcal{L}_A$ of all S -preclones on A is a **complete lattice** with intersection as meet and S -preclone generated by union as join.

Corollary

For a finite monoid S , the lattice ${}^S\mathcal{L}_A$ of S -preclones on a **finite** set A and the lattice of S -relational clones on A are **dually isomorphic**.

The lattice of S -preclones for a finite set A

Theorem

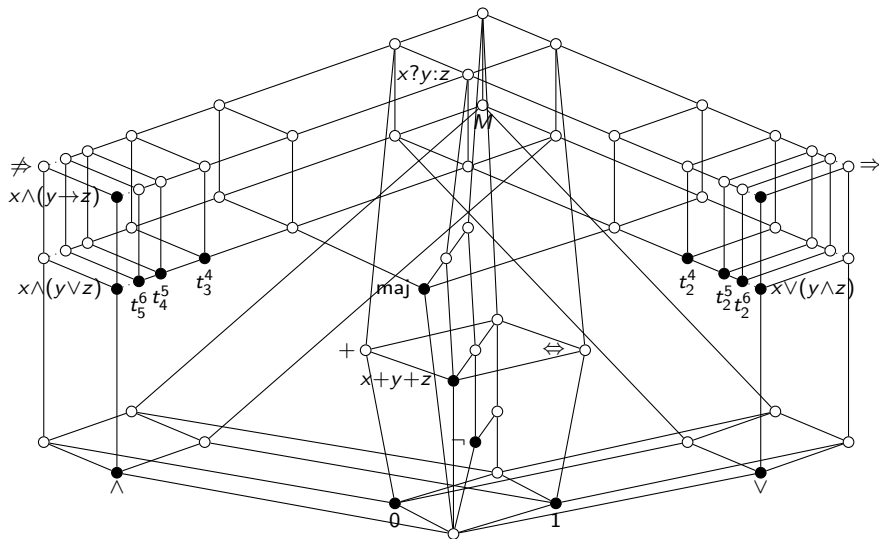
- The S -preclone ${}^S\text{Op}(A)$ is finitely generated.
- The S -relational clone ${}^S\text{Rel}(A)$ is finitely generated.
- The lattice ${}^S\mathcal{L}_A$ is atomic and coatomic.
- There are finitely many maximal and finitely many minimal S -preclones in ${}^S\mathcal{L}_A$.

If $A = \{0, 1\}$ then clones on A are called **Boolean clones**

The lattice $\mathcal{L}_{\{0,1\}}$ of Boolean clones is called the **Post lattice** since it was first described by **Emil Post [1941]**.

The Post lattice has **5 maximal** clones and **7 minimal** clones.

The Post lattice of all clones on a 2-element set



$$t_k^n(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } |\{i \mid x_i = 1\}| \geq k \\ 0 & \text{otherwise.} \end{cases}$$

All nine maximal Boolean \pm -preclones

Theorem

There are **nine** maximal Boolean \pm -preclones listed below. Each such preclone is of the form $F = {}^{\pm}\text{Pol } \rho$ for some \pm -relation $\rho = (\rho_+, \rho_-)$:

- a ${}^{\pm}\text{Pol}(\sigma, \sigma)$ with $\sigma \in \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ where $\text{Pol } \sigma_i$ is maximal in \mathcal{L}_2 (0-preserving, 1-preserving, monotone, self-dual, linear operations)
 $\sigma_0 = \{0\}$, $\sigma_1 = \{1\}$, $\sigma_2 = \leq = \{(0, 0), (0, 1), (1, 1)\}$,
 $\sigma_3 = \{(0, 1), (1, 0)\}$, $\sigma_4 = \{(x, y, z, u) \in A^4 \mid x + y + z + u = 0\}$.
- b ${}^{\pm}\text{Pol}(\leq, \geq) =$ all functions where each $+$ argument is order-preserving and each $-$ argument is order-reversing.
- c ${}^{\pm}\text{Pol}(A, \emptyset) =$ all functions with positive or mixed signum.
- d ${}^{\pm}\text{Pol}(A^2, \delta_A) =$ all Boolean \pm functions, where each negative argument is fictitious
- e ${}^{\pm}\text{Pol}(\{0\}, \{1\})$.

There are 20 minimal Boolean \pm -preclones

Theorem

There are twenty minimal Boolean \pm -preclones.

- A** $\pm\langle (x \wedge y) \vee (y \wedge z) \vee (z \wedge x) \rangle$, $\pm\langle x + y + z \rangle$
where the generators have signum $\lambda = (+, +, +, -)$.
- B** $\pm\langle h_0^\lambda \rangle$, $\pm\langle h_1^\lambda \rangle$, $\pm\langle h_y^\lambda \rangle$ where $h_i(x, y, z, u) = \begin{cases} x & \text{if } x = y \text{ or } z = u, \\ i & \text{otherwise,} \end{cases}$
and $\lambda = (+, +, -, -)$.
- C** $\pm\langle x \wedge y \rangle$, $\pm\langle x \wedge (y \vee z) \rangle$, $\pm\langle x \wedge (y \vee \neg z) \rangle$
 $\pm\langle x \vee y \rangle$, $\pm\langle x \vee (y \wedge z) \rangle$, $\pm\langle x \vee (y \wedge \neg z) \rangle$
 $\pm\langle (x \wedge y) \vee (y \wedge z) \vee (z \wedge x) \rangle$, $\pm\langle (x \wedge y) \vee (y \wedge \neg z) \vee (\neg z \wedge x) \rangle$
where the generators have signum $\lambda = (+, +, -)$
- D** $\pm\langle 0 \rangle$, $\pm\langle 1 \rangle$, $\pm\langle y \rangle$, $\pm\langle \neg y \rangle$, $\pm\langle \neg x \rangle$, $\pm\langle x \wedge y \rangle$, $\pm\langle x \vee y \rangle$
where the generators have signum $\lambda = (+, -)$

Some open problems

Is the lattice of Boolean \pm -preclones countable?

Classify the maximal S -preclones for $|S| > 2$ and $|A| > 2$.

Classify the maximal \pm -preclones below ${}^{\pm}\text{Pol}((\leq, \geq))$.

References



S. BONZIO, J. GIL-FÉREZ, P. JIPSEN, A. PRENOSIL, M. SUGIMOTO, **Balanced residuated partially ordered semigroups**, (2025)
<https://arxiv.org/abs/2505.12024>



P. JIPSEN, E. LEHTONEN, R. PÖSCHEL, **S-preclones and the Galois connection $\mathbf{SPol-SInv}$, Part I**, Algebra Univers. (2024) 85:34,
<https://doi.org/10.1007/s00012-024-00863-7>



P. JIPSEN, S. SANTSCI, **Residuated lattices do not have the amalgamation property**, (2025) <https://arxiv.org/abs/2504.16111>



D. PIGOZZI. **Partially ordered varieties and quasivarieties**. 2004,
https://www.researchgate.net/publication/250226836_partially_ordered_varieties_and_quasivarieties



R. PÖSCHEL AND L.A. KALUŽNIN, **Funktionen- und Relationenalgebren**. Deutscher Verlag der Wissenschaften, Berlin, 1979, Birkhäuser Verlag Basel, Math. Reihe Bd. 67, 1979.

THANKS!