On residuated (involutive) po-monoids as models of the Lambek Calculus and multiplicative linear logic

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The big picture

Proof Theory

Studies proof systems in specific logics

Syntax: Sequents $\Gamma \Rightarrow \Delta$ (hyper) sequent rules

Semantics: Consequence relations, proof nets, frames

Algebraic Logic Categorical proof theory

Algebra

Studies categories of algebras over specific theories

Syntax: Equations $s = t$ (quasi) identities, universal formulas

Semantics: Free algebras homomorphisms, (quasi) varieties

Universal algebra Partially ordered algebra

The big picture adjusted slightly

Proof Theory

Studies proof systems
in specific logics

Syntax: Sequents $\Gamma \Rightarrow \Delta$
(hyper) sequent rules

Semantics: Consequence
relations, proof nets, frames

Algebraic Logic
Categorical proof theory

Partially ordered Algebra

Studies categories of po-algebras
over specific theories

Syntax: Inequations $s \leq t$
Horn formulas, universal formulas

Semantics: Free algebras
homomorphisms, (quasi) varieties

Universal partially ordered algebra
(Enriched) category theory

Partially ordered algebras

A **po-algebra** is a partially ordered set with operations that are either order-preserving **or order-reversing** in each argument.

A **po-variety of po-algebras** is a class of similar po-algebras defined by equations **or inequations** [Pigozzi 2004]. (It's actually a strict Horn theory)

A **residuated partially ordered magma** or **rpo-magma** $\mathbf{A} = (A, \leq, \cdot, \backslash, /)$ is a partially-ordered set $(A, \leq)$ with a binary operation $\cdot$ and two **residuals** that satisfy for all $x, y, z \in A$

$$\text{(res)} \qquad xy \leq z \iff x \leq z/y \iff y \leq x\backslash z$$

The operation $x \cdot y$ is usually written xy.

A **residuated po-monoid** or **rpo-monoid** $\mathbf{A} = (A, \leq, \cdot, \backslash, /, 1)$ is an associative rpo-magma with an identity element 1.

Examples: **Lattice-free reducts** of residuated lattices are rpo-monoids.

Residuated po-magmas are a po-variety

Residuation ensures that x/y and $y\backslash x$ are **order-preserving** in the numerator (x position) and **order-reversing** in the denominator.

xy is order-preserving in both arguments.

(res) is equivalent to $\quad x \leq xy/y, \quad (z/y)y \leq z, \quad y \leq x\backslash xy, \quad x(x\backslash z) \leq z$

hence **rpo-magmas are a po-variety of po-algebras**.

Residuated lattices are also a po-variety, but $\leq$ is defineable by $x \wedge y = x$, so they are algebras.

rpo-monoids are models of the **Lambek Calculus**.

Involutive po-monoids

An **involutive po-monoid** (ipo-monoid) **A** is an algebraic structure $(A, \leq, \cdot, \sim, -, 0)$ such that $(A, \leq)$ is a poset, $\cdot$ is **associative** and

$$\text{(lin)} \qquad x \leq y \iff x \cdot {\sim}y \leq 0 \iff -y \cdot x \leq 0$$

Even without associativity (lin) has some interesting consequences:

Lemma

$-, \sim$ *form a dual Galois connection.*

Proof.

$-x \leq y \iff -x \cdot {\sim}y \leq 0 \iff {\sim}y \leq x.$ □

Hence $-, \sim$ are order reversing, $\quad -{\sim}x \leq x, \qquad {\sim}{-}x \leq x,$

${\sim}{-}{\sim}x = {\sim}x \quad$ and $\quad -{\sim}{-}x = -x.$

The linear negations are involutive

Lemma

$\sim - x = x = - \sim x$, *i.e., $\sim, -$ are involutions.*

Proof.

$x \leq - \sim x \iff x \cdot \sim - \sim x \leq 0 \iff x \cdot \sim x \leq 0 \iff x \leq x.$ □

For the next few results we use associativity.

Lemma

$\sim 0 = -0$ is an identity element.

Proof.

$-0 \cdot x \leq y \iff -0 \cdot x \cdot \sim y \leq 0 \iff x \cdot \sim y \leq 0 \iff x \leq y$. Hence $-0 \cdot x = x$

and similarly $x \cdot \sim 0 = x$. Therefore $\sim 0 = -0 \cdot \sim 0 = -0$. □

Naturally ~ 0 is denoted by 1.

Multiplication is residuated, hence order-preserving

Lemma

$xy \leq z \iff x \leq -(y \cdot {\sim}z) \iff y \leq {\sim}(-z \cdot x).$

Proof.

$xy \leq z \iff (-{\sim}x)y{\sim}z \leq 0 \iff y{\sim}z \leq {\sim}x \iff x \leq -(y{\sim}z).$
Similarly $xy \leq z \iff -zx({\sim}-y) \leq 0 \iff -zx \leq -y \iff y \leq {\sim}(-zx).$ □

Therefore $z/y = -(y \cdot {\sim}z)$ and $x\backslash z = {\sim}(-z \cdot x)$.

Lemma

$x \leq y \implies xz \leq yz$ *and* $zx \leq zy$.

Proof.

Follows from residuation: $yz \leq yz \implies x \leq y \leq yz/z \implies xz \leq yz.$ □

Algebraic models of multiplicative linear logic (MLL)

The previous results show that ipo-monoids are po-algebraic models of the **non-commutative, non-cyclic multiplicative linear logic** ($x + y = -({\sim}y \cdot {\sim}x) = {\sim}(-y \cdot -x)$ and $x^{\perp} = {\sim}x$, ${}^{\perp}x = -x$).

With **cyclicity** ($xy \leq 0 \Leftrightarrow yx \leq 0$) added, we obtain models of **cyclic multiplicative linear logic** (CyMLL), where ${\sim}x = -x$.

With commutativity ($xy \leq yx$) added, we obtain models of the **multiplicative linear logic**. MLL entailment has FMP and is decidable (NP-complete!).

Some of these algebras (reducts of distributive commutative involutive residuated lattices) are models of **multiplicative fragment of contraction-free relevance logic** (RW_m^t).

We will see that ipo-monoids come in a form of **group**-indexed unions of directed (bounded) connected components $\bigcup_{g \in G} A_g$. Depending on additional axioms or assumptions on the unital component A_e, we can say a bit more.

Examples: binary relations, pointed groups, po-groups

Any set of binary relations $A \subseteq \mathcal{P}(X^2)$ including id_X, closed under composition and **complement-converse** $\sim R = -R = X^2 \setminus R^{-1}$ is an **ipo-monoid**.

Weakening relations on a poset with composition and complement-converse.

A **pointed group** $(A, \cdot, ^{-1}, 1, 0)$ is a group with an (arbitrary) constant 0.

$\Rightarrow$ obtain an **ipo-monoid** with $\leq$ be equality, $\sim x = x^{-1} \cdot 0$ and $-x = 0 \cdot x^{-1}$.

A **partially ordered group** is a structure $\mathbf{A} = (A, \leq, \cdot, ^{-1}, 1)$ such that

1. $(A, \leq)$ is a poset
2. $(A, \cdot, ^{-1}, 1)$ is a group and
3. multiplication is order preserving (hence $^{-1}$ is order reversing).

$\Rightarrow$ obtain a cyclic ipo-monoid with $0 = 1^{-1} = 1$ and $\sim x = -x = x^{-1}$.

Examples: $(\mathbb{Z}, \leq, +, -, 0)$, $(\mathbb{R}, \leq, +, -, 0)$, $(\mathbb{Q}^+, \leq, \cdot, ^{-1}, 1)$,
all groups (with $\leq$ as $=$, e.g. $(\mathbb{Z}, =, +, -, 0)$)

Lattice-free reducts of involutive residuated lattices

An **involutive residuated lattice** $(A, \wedge, \vee, \cdot, \sim, -, 0)$ is a lattice $(A, \wedge, \vee)$ such that $(A, \leq, \cdot, \sim, -, 0)$ is an ipo-monoid.

Hence **lattice-free** reducts of involutive residuated lattices are ipo-monoids.

Recall that **IMTL-algebras** are commutative ($xy = yx$) integral ($x \leq 1$) involutive prelinear ($1 \leq x/y \vee y/x$) residuated lattices, and

MV-algebras are divisible ($x \wedge y = (x/y) \cdot y$) IMTL-algebras

$\Rightarrow$ **MV-algebras are ipo-monoids** (using only $\cdot, -, 0$ since $x \vee y = -(-x \wedge -y)$).

Sugihara ipo-monoids (reducts of algebraic models of RM_m)

$\mathbf{S}_{2n}$: a $2n$-element chain $\quad a_n < \cdots < a_2 < \mathbf{0} < \mathbf{1} < b_2 < \cdots < b_n$

$\mathbf{S}_{2n-1}$: a $2n$–1-element chain $\quad a_n < \cdots < a_2 < \mathbf{0} = \mathbf{1} < b_2 < \cdots < b_n$.

Define $\quad a_1 = 0, \qquad b_1 = 1, \qquad {\sim}a_i = -a_i = b_i, \qquad {\sim}b_i = -b_i = a_i,$

$$a_i \cdot a_j = a_{\max(i,j)}, \qquad b_i \cdot b_j = b_{\max(i,j)}, \qquad a_i \cdot b_j = b_j \cdot a_i = \begin{cases} b_j & \text{if } i < j \\ a_i & \text{otherwise} \end{cases}$$

$(\mathbf{S}_m, \leq, \cdot, \sim, -, 0)$ is the ***m*-element Sugihara ipo-monoid** (the m-element lattice-free reduct of the Sugihara monoid)

$\mathbf{S}_2 = \mathbf{2}$ is the two-element **Boolean algebra** $\mathbf{2} = (\{0, 1\}, \wedge, \neg, \neg, 0)$

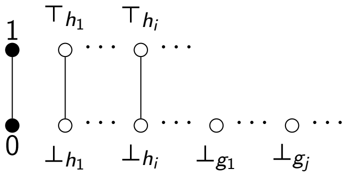
$\mathbf{S}_3$ is also known as the **Gaifman-Pratt ipo-monoid**

Models from products of groups and Boolean algebras

Let $\mathbf{G}$ be a group with subgroup $\mathbf{H}$.

Define $\mathbf{2_{G,H}} = \mathbf{H} \times \mathbf{2} \cup (\mathbf{G} \setminus \mathbf{H}) \times \mathbf{1}$ with

$(g, a) \cdot (g', a') = (gg', aa')$ and $\sim(g, a) = (g^{-1}, \sim a) = -(g, a)$.



Then $\mathbf{A}$ is an ipo-monoid. $\mathbf{A}$ is a **conuclear image** of $\mathbf{G} \times \mathbf{2}$ (the conucleus is the map $(g, x) \mapsto (g, \top_g \cdot \top_{g^{-1}} \cdot x)$).

Conuclei on rpo-monoids and ipo-monoids

A **conucleus** δ on an rpo-monoid $\mathbf{A}$ is a **conuclear interior** operator:

- $\delta(x) \cdot \delta(y) \leq \delta(xy)$, and $\delta(x)\delta(1) = \delta(x) = \delta(1)\delta(x)$ (conuclear)
- $x \leq y \implies \delta(x) \leq \delta(y)$, and $\delta(\delta(x)) = \delta(x) \leq x$ (interior)

The **conuclear image** of $\mathbf{A}$ is $\mathbf{A}_\delta = (\delta[A], \leq, \cdot, \backslash_\delta, /_\delta, \delta(1))$,

where $x \backslash_\delta y = \delta(x \backslash y)$ and $x /_\delta y = \delta(x/y)$.

Lemma

(a) $\mathbf{A}_\delta$ *is an rpo-monoid.*

(b) *If* $\mathbf{A}$ *is an ipo-monoid and* δ *satisfies*

- $\delta(-(\delta({\sim}\delta(x)))) \leq \delta(x)$ *and* $\delta({\sim}(\delta(-\delta(x)))) \leq \delta(x)$

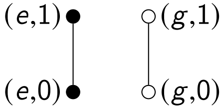
then $\mathbf{A}_\delta = (\delta[A], \leq, \cdot, \delta(0), \sim_\delta, -_\delta)$ *is an ipo-monoid,*

where ${\sim}_\delta x = \delta({\sim}x)$ *and* $-_\delta x = \delta(-x)$.

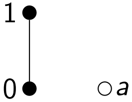
$\mathbf{2}_{\mathbb{Z}_2,\{e\}}$ and $\mathbb{Z}_2 \times \mathbf{2}$

One of the simplest examples of ipo-monoids $\mathbf{2}_{\mathbb{Z}_2,\{e\}}$ arises as a conuclear image of the product algebra $\mathbb{Z}_2 \times \mathbf{2}$:

$\mathbb{Z}_2 \times \mathbf{2}$



$\mathbf{2}_{\mathbb{Z}_2,\{e\}}$



(black dots denote idempotent elements)

$a = {\sim}a$ acts as an infectious value, but $a^2 = 0$, and $a^3 = a$.

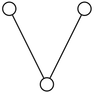
We will later discuss inequational bases for the (po-)varieties generated by $\mathbf{2}_{\mathbb{Z}_2,\{e\}}$ and by $\mathbb{Z}_2 \times \mathbf{2}$.

The posets of residuated po-magmas

Up to isomorphism, there are **many** rpo-magmas:

cardinality $n =$	1	2	3	4
number of rpo-magmas	1	3	28	1200

Although rpo-magmas are very general, (res) imposes restrictions on the posets that can occur.

For example, could   be the poset of a rpo-magma? (Hint: no)

Lemma (1)

For rpo-magmas, if $a, b \leq c$ *then* $(a/(a\backslash c))((c/(a\backslash c))\backslash b) \leq a, b$*.*

Proof.

$$\begin{array}{rll}
\text{Assume } a \leq c. & \text{Then} & a/(a\backslash c) \leq c/(a\backslash c) \\
\Longrightarrow & & (c/(a\backslash c))\backslash b \leq (a/(a\backslash c))\backslash b \\
\Longleftrightarrow & & (a/(a\backslash c))((c/(a\backslash c))\backslash b) \leq b \\
\text{Assume } b \leq c. & \text{Then} & a\backslash c \leq a\backslash c \\
\Longleftrightarrow & & a(a\backslash c) \leq c \\
\Longleftrightarrow & & a \leq c/(a\backslash c) \\
\Longrightarrow & & (c/(a\backslash c))\backslash b \leq a\backslash b \leq a\backslash c \\
\Longrightarrow & & a/(a\backslash c) \leq a/((c/(a\backslash c))\backslash b) \\
\Longleftrightarrow & & (a/(a\backslash c))((c/(a\backslash c))\backslash b) \leq a
\end{array}$$

$\square$

Finite rpo-magmas have bounded components

Lemma (2; proof is similar)

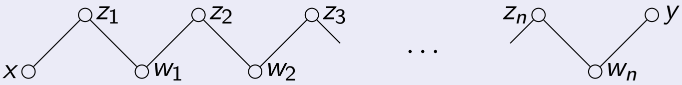
In any rpo-magma, if $d \leq a, b$ then
$a, b \leq d/(((a\backslash d)/(a\backslash(d\backslash d)))((d\backslash d)/(a\backslash(d\backslash d)))\backslash(b\backslash d)))$

Theorem

In an rpo-magma every connected component of $\leq$ is up-directed and down-directed, hence for ***finite*** *rpo-magmas every connected component is* ***bounded****.*

Proof.

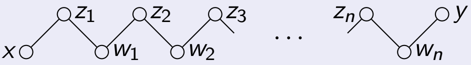
Two elements x, y in a poset are connected iff there exists a zigzag



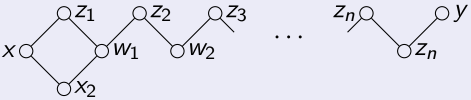
We need to find an upper and a lower bound of x, y.

□

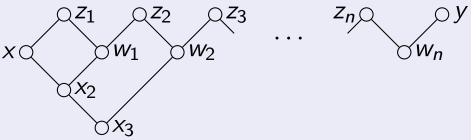
Proof (continued).



Note that $x, w_1 \leq z_1$, so apply Lemma (1) to obtain $x_2 \leq x, w_2$:



Now $x_2, w_2 \leq z_2$, so apply Lemma (1) again to obtain $x_3 \leq x_2, w_2$:



Use induction to get a lower bound of x, y in n steps.

Similarly use Lemma (2) to obtain an upper bound. □

What posets occur for rpo-magmas?

In a poset define $x \leftrightsquigarrow y$ if x and y are in the same connected component.

A **quasigroup** $\mathbf{Q} = (Q, \cdot, \backslash, /)$ is a set with 3 binary operations such that

$$xy = z \iff x = z/y \iff y = x \backslash z.$$

Theorem

*For any po-algebra the relation $\leftrightsquigarrow$ is a **congruence**.*

*For a rpo-magma $\mathbf{A}$ the quotient algebra $\mathbf{A}/\leftrightsquigarrow$ is a **quasigroup** with $\leq$ as equality.*

*Conversely, from any quasigroup $\mathbf{Q}$ and a pairwise disjoint family of **bounded** posets A_g for $g \in \mathbf{Q}$, one can construct an rpo-magma with poset $\bigcup_{g \in \mathbf{Q}} A_g$.*

For example, for a quasigroup $\mathbf{Q}$ and $x \in A_g$, $y \in A_h$ define

$$x \cdot y = \bot_{gh}, \qquad x \backslash y = \top_{g \backslash h}, \qquad x/y = \top_{g/h}.$$

Residuated po-semigroups

A **rpo-semigroup** or **Lambek algebra** is a rpo-magma where $\cdot$ is associative. These are algebraic models of Lambek Calculus.

Lemma

Any rpo-semigroup for which the poset is an antichain, is a ***group****.*

Proof.

An associative quasigroup is a group: $x = y\backslash yx = y(y\backslash x)$.

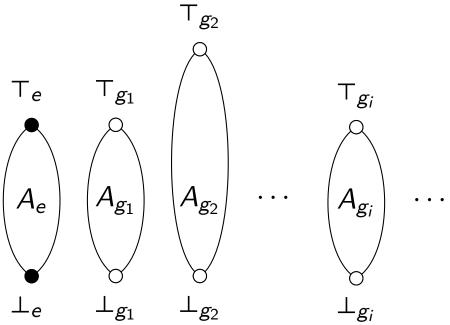
Hence $x = x(x\backslash x) \Rightarrow xy = x(x\backslash x)y \Rightarrow y = x\backslash xy = (x\backslash x)y \Rightarrow y/y = x\backslash x$.

So $x\backslash x = e$ is constant, $xe = x$ and $ey = y$, i.e., e is an identity.

Now $(e/x)x = e \Rightarrow x(e/x)x = x \Rightarrow x(e/x) = x/x = e$, so $x^{-1} = e/x$. $\square$

$\Rightarrow$ For any rpo-semigroup $\mathbf{A}$ (and therefore **for any ipo-monoid**) the quotient $\mathbf{A}/\!\!\rightsquigarrow$ is a **group**.

Poset of a rpo-semigroup with a lower bound somewhere



Lemma

If any connected component is lower bounded, they are all bounded.

Unital rpo-monoids with trivial unital connected component

The connected component that contains the identity element is denoted by A_e.

Lemma

Suppose $\mathbf{A}$ *is a rpo-magma with an identity element* 1.

If $A_e = \{1\}$ *then* $\mathbf{A}$ *is a quasigroup with antichain order.*

Proof.

Assume $A_e = \{1\}$. Then $z \leq 1 \Rightarrow 1 \leq z$ and it suffices to show $x \leq y \Rightarrow y \leq x$.

From (res), $1 \leq x/x$ implies $1/(x/x) \leq (1/(x/x)) \cdot (x/x) \leq 1$.

Since 1 is minimal, we have $1 \leq 1/(x/x)$, hence $x/x \leq 1$.

Now assume $x \leq y$. Then $(x/y)x \leq (x/y)y \leq x$. By residuation $x/y \leq x/x \leq 1$.

Again from minimality of 1, we deduce $1 \leq x/y$, hence $y \leq x$. $\square$

It follows that any rpo-monoid with $A_e = \{1\}$ is a **group** with the antichain order, and any ipo-monoid with $A_e = \{1\}$ is a **pointed group** with the antichain order.

Ipo-monoids with additional assumptions

The connected components of $\mathbf{A}$ are denoted A_g for $g \in \mathbf{A}/\leftrightsquigarrow$, with the **unital component** $1 \in A_e$. All components are up-and-down-directed.

The **order** of $x \in \mathbf{A}$ is the smallest positive $n \in \mathbb{N}$ s.t. $x^n \in A_e$, or ∞ if no n exists.

If $\mathbf{A}/\leftrightsquigarrow$ is finite and every element has a finite order, they all divide the order of the group, so $\exists n \forall x (x^n \leftrightsquigarrow 1)$. We are interested in the map $f : A_g \to A_e$, $f(x) = x^n$, whether it is an order embedding and what is its image.

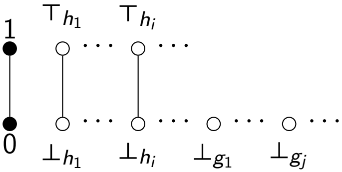
In particular we look at

- Ipo-monoids where $0 \leq 1$, and A_e is idempotent
- Ipo-monoids with $x^{n+1} = x$ for some n (**n-periodic contraction**)
- Ipo-monoids where $x \leq 0$ or $1 \leq x$ for all $x \in A_e$
- **weakly integral** ipo-monoids ($1 \leq x \Rightarrow 1 = x$)
- Ipo-monoids where A_e is an ipo-**Sugihara monoid**, or a **Boolean algebra**.

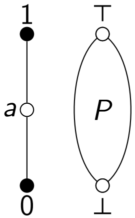
Ipo-monoids with specified bounded unital component

$A_e = \{1\}$ implies $\mathbf{A}$ is a group

$A_e = \{0, 1\} = \mathbf{2}$ implies all components have 1 or 2 elements, hence we get $\mathbf{2}_{\mathbf{G},\mathbf{H}}$



$A_e = \{0 < a < 1\} = \mathbf{L}_3$ with $a^2 = 0$
$\mathbf{G} = \mathbb{Z}_2$, P is any self-dual poset



$$x \cdot y = \begin{cases} 0 & \text{if } x \le -y \\ x & \text{if } x \ne 0, y = 1 \\ y & \text{if } x = 1, y \ne 0 \\ a & \text{if } x, y \in P, x \not\le -y \\ \bot & \text{if } x \in \{0, a\}, y \in P \\ \bot & \text{if } x \in P, y \in \{0, a\} \end{cases}$$

What about cyclicity/commutativity?

- Ipo-monoids of unions of **finite chains** with $0 \leq 1$ are **cyclic** (i.e. $\sim x = -x$).
- **Cyclic** ipo-monoids with a **Boolean** A_e are **commutative** if they satisfy $x^{n+1} = x$ for some $n \geq 2$. Not all cyclic ipo-monoids with a Boolean A_e are commutative.
- If (every element of **A** has a finite order), $A_e = \{x \mid x \leq 0 \text{ or } 1 \leq x\}$, $0 \leq 1$ and A_e is idempotent, then
  - A_e is cyclic if and only if **A** is **balanced** ($x \cdot \sim x = -x \cdot x =: 0_x$ and $x \backslash x = x / x =: 1_x$)
  - If A_e is cyclic then A_e is a commutative subalgebra of **A**,
  - If A_e is cyclic and A is finite, then **A** is cyclic.
- Ipo-monoids with a **Sugihara** $A_e = S_m$ are all **cyclic** (and therefore balanced).
- There are finite single-component non-cyclic ipo-monoids (6 elements), even chains (7 elements).

A non-cyclic ipo-monoid

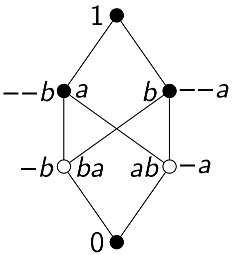


Figure: Smallest noncyclic ipo-monoid with a single connected component.

Ipo-monoids where $0 \leq 1$ (A_e is a subalgebra of $\mathbf{A}$)

Lemma

In an ipo-monoid with $0 \leq 1$, *for any* n, $(\sim x)^n \leq \sim x^n$ *and* $(-x)^n \leq -x^n$.

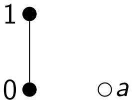
Proof.

By induction, assume $(\sim x)^n \leq \sim x^n$. Then $x^n(\sim x)^n \leq 0 \leq 1$.

Now $x \cdot x^n(\sim x)^n \cdot \sim x \leq x \cdot \sim x \leq 0$.

Hence $x^{n+1}(\sim x)^{n+1} \leq 0$ and therefore $(\sim x)^{n+1} \leq \sim x^{n+1}$. □

The converse inequality $\sim x^n \leq (\sim x)^n$ is not true in general: it fails for $x = a$ and $n = 2$ in the following three-element ipo-monoid with $a = \sim a$ acting as an infectious value, and $a^2 = 0$ and $a^3 = a$:



Construction of Płonka sum of two ipo-monoids

Let $\mathbf{A}, \mathbf{B}$ be disjoint involutive residuated lattices or ipo-monoids, and $\varphi : \mathbf{A} \to \mathbf{B}$ a monoid homomorphism such that φ preserves existing joins (hence preserves the order) and $\varphi(x) \neq 0^{\mathbf{B}}$ for all $x \in A$.

On $A \cup B$, define $\sim\ =\ \sim^{\mathbf{A}} \cup \sim^{\mathbf{B}}$, $- = -^{\mathbf{A}} \cup -^{\mathbf{B}}$,

$$x \cdot y = \begin{cases} x \cdot^{\mathbf{A}} y & \text{if } x, y \in A, \\ x \cdot^{\mathbf{B}} y & \text{if } x, y \in B, \\ \varphi(x) \cdot^{\mathbf{B}} y & \text{if } x \in A \text{ and } y \in B, \\ x \cdot^{\mathbf{B}} \varphi(y) & \text{if } x \in B \text{ and } y \in A, \end{cases} \qquad x \leq y \iff x \cdot {\sim} y = 0^{\mathbf{A}}$$

and let $\mathbf{A} \uplus_{\varphi} \mathbf{B} = (A \cup B, \leq, \cdot, \sim, -, 0^{\mathbf{A}})$ be the **binary Płonka sum**.

Theorem

For any ipo-monoids $\mathbf{A}, \mathbf{B}$ the po-algebra $\mathbf{A} \uplus_{\varphi} \mathbf{B}$ is an ipo-monoid.

Decomposition into Płonka sum of two ipo-monoid

The term 1_x is defined as $-(x \cdot {\sim}x) = x/x$.

An ipo-monoid is **balanced** if $x/x = x\backslash x$ and **steady** if $1_x \cdot 1_y = 1_{xy}$.

Theorem

Let $\mathbf{C}$ *be a balanced steady involutive residuated lattice or ipo-monoid and assume* (p, q) *is a splitting pair of* ***central idempotent*** *elements in* $\mathbf{C}^+$*, i.e.,* $\mathbf{C}^+ = {\downarrow}p \uplus {\uparrow}q$.

Let $A = \{x \in C \mid 1_x \leq p\}$, $B = \{x \in C \mid q \leq 1_x\}$ *with restricted operations from* $\mathbf{C}$.

Then $\mathbf{A}$ *is a subalgebra,* $\mathbf{B}$ *is a 1-free subalgebra of* $\mathbf{C}$ *with identity* q *and* $C = A \uplus B$.

Define $\varphi : A \to B$ *by* $\varphi(x) = x \cdot q$.

Then φ *is a monoid homomorphism that preserves all existing joins and* $\mathbf{C} = \mathbf{A} \uplus_\varphi \mathbf{B}$.

Commutative ipo-monoids where $0 \leq 1$ and A_e idempotent

Lemma

Let **A** *be a commutative ipo-monoid with a unital component that is idempotent. Then*

- *A_e is a locally-integral ipo-monoid. Positive elements A^+ of A_e form a join-semilattice.*
- **A** *is steady:* $1_x \cdot 1_y = 1_{xy}$

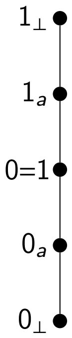
Proof (sketch).

A is commutative, and therefore cyclic and balanced. The units $1_x = x\backslash x \geq 1$ are positive, $1_x \cdot x = x$. If $x \in A_e$, then $x \leq 1_x$ and $x\backslash 1_x = 1_x$ by idempotence.
By Thm. 5.1 of the paper below, **fibers $A_{1_x} = \{y \mid 1_x = 1_y\}$ of A_e are Boolean**.
Proof of steadiness uses commutativity in an essential way. □

A_e can be constructed as a **Płonka sum** of its fibers A_p, indexed by the join semilattice A^+. **Which ipo-monoids A can we construct using Płonka sums?**

J. Gil-Férez, P. Jipsen, M. Sugimoto: Locally integral involutive PO-semigroups, Fundamenta Informaticae, Volume 195, 1-4, 7, (2025) 1–29

Example: $\mathbf{S}_5$



$$\mathbf{S}_5 = \mathbf{1} \uplus_{\varphi_1} \mathbf{2} \uplus_{\varphi_2} \mathbf{2}$$

Commutative ipo-monoids with Boolean unital component

Lemma

Let **A** *be a commutative ipo-monoid with a unital component that is a Boolean algebra* **B** *and denote the group* $\mathbf{A}/\!\rightsquigarrow$ *by* **G**. *Then* **A** *is a conuclear image of* $\mathbf{G} \times \mathbf{B}$ *and all components of* **A** *are isomorphic to principal ideals of* **B**.

Proof (sketch).

The conuclear map on $\mathbf{G} \times \mathbf{B}$ is defined by $\delta((g, x)) = (g, \top_g \cdot \top_{g^{-1}} \cdot x)$. The proof uses that $\top_g \cdot \top_{g^{-1}} \cdot \top_g = \top_g$. □

In case every element of **A** has a finite order, we obtain the following insight: let order of $x, y \in A_g$ be n. Then

- $x^n = y^n$ iff $x = y$, and
- $x = x^{n+1} = x \cdot \top_g \cdot \top_{g^{-1}}$.

We will use **conuclear images of $\mathbf{G} \times \mathbf{B}$** as **fibers** of **Płonka sums** to construct commutative ipo-monoids (e.g. those with Sugihara unital component).

Ipo-monoids where $0 \leq 1$, and $x \leq 0$ or $1 \leq x$ for all $x \in A_e$

Note that the assumptions hold if $A_e \simeq \mathbf{S}_k$.

Lemma

Let A be an ipo-monoid such that $0 \leq 1$ and $(x \leq 0$ or $1 \leq x)$ for all $x \in A_e$.

If $x, y \in A_g$ have order n then $f : A_g \to A_e$ given by $f(x) = x^n$ is an order embedding.

Proof.

Assume $x^n \leq y^n$. We want to show $x \leq y$.

Suppose $x \not\leq y$, then $x \cdot {\sim}y \not\leq 0$. Note $x, y \in A_g \implies x \cdot {\sim}y \in A_e$.

From $(z \leq 0$ or $1 \leq z)$ for $z \in A_e$ we conclude $1 \leq x \cdot {\sim}y$.

Hence $1 \leq x \cdot {\sim}y \leq x \cdot 1 \cdot {\sim}y \leq x \cdot x \cdot {\sim}y \cdot {\sim}y \leq \cdots \leq x^n \cdot ({\sim}y)^n$.

By the preceeding lemma and $x^n \leq y^n$, $x^n \cdot ({\sim}y)^n \leq x^n \cdot {\sim}y^n \leq 0$.

Therefore $x \cdot {\sim}y \leq 0$, a contradiction. $\square$

Structure of ipo-monoids with $A_e \cong \mathbf{S}_k$

Theorem

Let $\mathbf{A}$ be an ipo-monoid with $A_e \cong S_k$ where every element is of a finite order. Then $\mathbf{A}$ is a disjoint union of chains, each with $\leq k$ elements, and $\cdot$ determined by subgroups of $\mathbf{A}/\!\rightsquigarrow$. Moreover, $\mathbf{A}$ is cyclic and balanced.

Proof (outline).

Let $\mathbf{G} = \mathbf{A}/\!\rightsquigarrow$. The map $f : A_g \to A_e$ is an **order embedding.** Therefore all connected components are chains with $\leq k$ elements.

For $1 \leq l \leq k$, let $\mathbf{H}_l = \{g \in \mathbf{G} \mid l \leq |A_g|\}$.

Then $\mathbf{H}_k \leq \cdots \leq \mathbf{H}_2 \leq \mathbf{H}_1 = \mathbf{G}$ is a nested sequence of subgroup of $\mathbf{G}$ and these subgroups uniquely determine the multiplication on the algebra. $\square$

This result holds without assuming finite order, if A is balanced.

Structure of ipo-monoids with $A_e \cong \mathbf{S}_k$

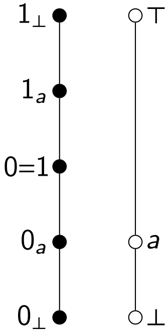


Figure: The ipo-monoid $\mathbf{1} \uplus_{\varphi_1} \mathbf{2}_{\mathbb{Z}_2,\{e\}} \uplus_{\varphi_2} (\mathbb{Z}_2 \times \mathbf{2})$ with S_5 as unital component

Structure of commutative ipo-monoids with $A_e \cong \mathbf{S}_k$

Every **commutative** ipo-monoid $\mathbf{A}$ with $A_e \cong \mathbf{S}_k$ can also be constructed using a **Płonka sum** over

- a linear join-semilattice of positive elements A^+ with the bottom element 1,
- A^+ directed system of ipo-monoids $A_{1_x} = \{y \mid 1_y = 1_x\}$ (fibers)
- $|A_{1_x} \cap A_g| \leq 2$
- fibers are ipo-monoids of the form $\mathbf{2}_{\mathbf{G},\mathbf{H}}$.

$\Longrightarrow$ **Every commutative ipo-monoid A with $A_e \cong \mathbf{S}_k$ and $\mathbf{G} = \mathbf{A}/\!\!\leftrightsquigarrow$ is a conuclear image of $\mathbf{G} \times \mathbf{S}_k$** (due to X. Zhuang for odd k).

J. Gil-Férez, P. Jipsen, M. Sugimoto: Locally integral involutive PO-semigroups, Fundamenta Informaticae, Volume 195, 1-4, 7, (2025) 1–29.

S. Bonzio, J. Gil-Férez, P. Jipsen, A. Přenosil, M. Sugimoto: On the structure of balanced residuated partially ordered monoids, RAMiCS 2024, LNCS Vol 14787, Springer 2024, 83–100.

X. Zhuang: Unilinear residuated lattices, (2023) Dissertation, digitalcommons.du.edu/etd/2335/

Structure of commutative ipo-monoids with $A_e \cong \mathbf{S}_k$ - Płonka sums

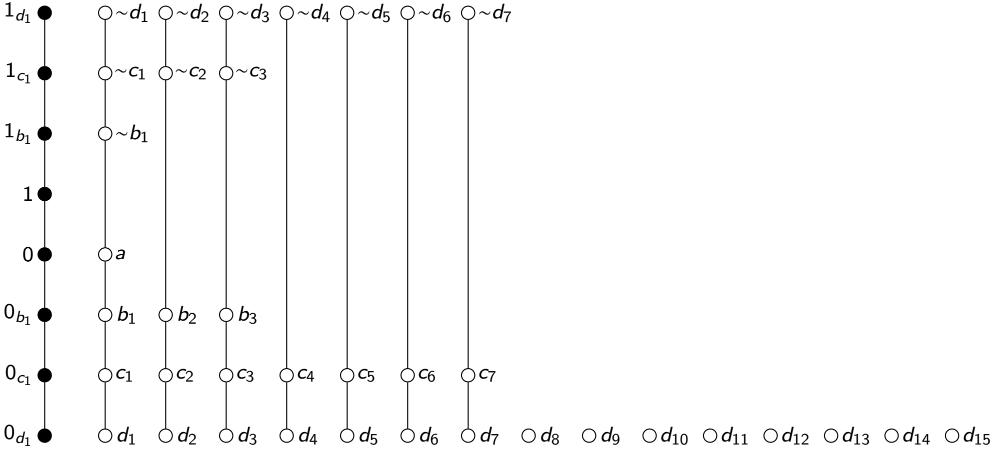
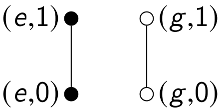


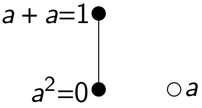
Figure: Example of a Płonka sum $\mathbf{2}_{\mathbb{Z}_2,\{e\}} \uplus_{\varphi_1} \mathbf{2}_{\mathbb{Z}_2^2,H_1} \uplus_{\varphi_2} \mathbf{2}_{\mathbb{Z}_2^3,H_2} \uplus_{\varphi_3} \mathbf{2}_{\mathbb{Z}_2^4,H_3}$

$\mathbb{Z}_2 \times \mathbf{2}$ and $\mathbf{2}_{\mathbb{Z}_2,\{e\}}$-generated (po-)varieties

$\mathbb{Z}_2 \times \mathbf{2}$



$\mathbf{2}_{\mathbb{Z}_2,\{e\}}$



The order is definable: $x \leq y \iff 1 = x \backslash y$.

The term $\tau(x) := {\sim}x^2 \cdot (x + x)$ equals 1 if $x = a$, and equals 0 otherwise.

(quasi-)inequational bases (CONJECTURE) for the varieties generated by

- $\mathbb{Z}_2 \times \mathbf{2}$
  - Boolean axioms instances in $(\cdot, +, \sim, 0)$ for x^2
  - ipo-monoid axioms
  - $x^3 = x$
  - ${\sim}(x^2) = ({\sim}x)^2$
- $\mathbf{2}_{\mathbb{Z}_2,\{e\}}$
  - Boolean axioms instances in $(\cdot, +, \sim, 0)$ for x^2
  - ipo-monoid axioms
  - $x^3 = x$
  - $\tau(x) \cdot \tau(y) \leq \tau(x) \cdot ({\sim}(xy))^2$

Finally, here's a question for Simon (and anyone else that is interested):

Do (commutative) rpo- or ipo-monoids have the **amalgamation property**?

Thank you!

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